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Infinite reduction of couplings in non-renormalizable quantum field theory

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Abstract: I study the problem of renormalizing a non-renormalizable theory with a reduced, eventually finite, set of independent couplings. The idea is to look for special relations that express the coefficients of the non-renormalizable terms as unique functions of a reduced set of independent couplings λ, such that the divergences are removed by means of field redefinitions plus renormalization constants for the λs. I consider non-renormalizable theories whose renormalizable subsector $\mathcal{R}$ is interacting. The "infinite" reduction is determined by i) perturbative meromorphy around the free-field limit of $\mathcal{R}$, or (ii) analyticity around the interacting fixed point of $\mathcal{R}$. In general, prescriptions (i) and (ii) mutually exclude each other. When the reduction is formulated using (i), the number of independent couplings remains finite or slowly grows together with the order of the expansion. The growth is slow in the sense that a reasonably small set of parameters is sufficient to make predictions up to very high orders. Instead, in case (ii) the number of couplings generically remains finite. The infinite reduction is a tool to classify the non-renormalizable interactions and address the problem of their physical selection.

Keywords: Renormalization Regularization and Renormalon, Renormalization Group.

Contents

1. Introduction

Fundamental theories should be able, at least in principle, to describe arbitrarily high energies with a finite number of independent couplings. The usual formulation of non-renormalizable theories makes use of infinitely many independent couplings to subtract the divergences. So formulated, non-renormalizable theories are good only as effective field theories, finitely many parameters being sufficient to make predictions about low-energy phenomena.

These facts, however, do not imply that non-renormalizable theories are useless or inadequate as fundamental theories, but only that their naive formulation is. An improved formulation should be uncovered. Some steps in this direction have been made in ref.s [1–3]. In [1, 2] finite and quasi-finite non-renormalizable theories have been constructed as irrelevant deformations of interacting conformal field theories. In [3] a certain class of non-renormalizable theories with a running renormalizable subsector $\mathcal{R}$ have been studied, in a perturbative framework of new type, which allows to treat unexpanded functions of the fields, although not of their derivatives. In some of the models of [3], with non-analytic potentials, the divergences are reabsorbed with a finite number of independent couplings. In more standard models the number of independent couplings grows together with the order of the expansion, but a certain form of predictivity is retained.

The purpose of this paper is to generalize the results of [3], develop the systematics of the "reduction of couplings" for non-renormalizable theories and study the predictive power of the reduced theories. The reduction of couplings is the search for special, unambiguous and self-consistent relations among the couplings, such that the lagrangian depends on a reduced, eventually finite, set of couplings λ and all divergences are removed by means of field redefinitions plus renormalization constants for the λs. The reduction is a tool to "diagonalize", and therefore classify, the non-renormalizable interactions. The potential applications of this investigation are to physics beyond the Standard Model and quantum gravity.

Unless otherwise specified, the words "relevant", "marginal" and "irrelevant" refer to the gaussian fixed point, so they are equivalent to "super-renormalizable", "strictly renormalizable" and "non-renormalizable", respectively. In the study of deformations of interacting conformal field theories, the construction of this paper allows also to characterize the deformation as marginal, relevant or irrelevant at the interacting fixed point.

The power-counting renormalizable sector $\mathcal{R}$ needs to be fully interacting, by which I mean that all marginal interactions are turned on. Indeed, the infinite reduction does not work when the marginal sector is free or only partially interacting. Without loss of generality, I assume also that $\mathcal{R}$ does not contain relevant couplings. This assumpion ensures that the beta functions depend polynomially on the irrelevant couplings. When $\mathcal{R}$ is fully interacting, relevant parameters can be added perturbatively *after* the construction of the irrelevant deformation.

The inclusion of relevant parameters with a free or only partially interacting renormalizable sector $\mathcal{R}$ is important for applications to quantum gravity in four dimensions, which has no marginal coupling, its relevant parameter being by the cosmological constant. However, further insight is needed to deal with the technical complicacies of this problem, so its investigation is postponed. Basically, in the constructions of this paper the interactions have to be turned on in the following order: first the marginal interactions, then the irrelevant interactions, finally the relevant interactions.

Denote the marginal couplings of $\mathcal{R}$ with α and the irrelevant couplings of the complete theory with λ_n. The subscript n denotes the "level" of λ_n, $-n$ being the dimensionality of λ_n in units of mass. The beta functions of the irrelevant couplings have the structure

$$\beta_{\lambda_n}(\alpha, \lambda) = \gamma_n(\alpha)\,\lambda_n + \delta_n(\lambda_{m<n}, \alpha)\,, \tag{1.1}$$

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where $\delta_n(\lambda_{m<n}, \alpha)$ is polynomial, at least quadratic, in the irrelevant couplings λ_m with $m < n$. The structure (1.1) is obtained matching the dimensionalities of the left- and right-hand sides. Indeed, in perturbation theory only integer powers of the couplings can appear and by assumption there are no couplings with positive dimensionalities in units of mass. Therefore β_{λ_n} is at most linear in λ_n, polynomial in the irrelevant couplings λ_k with $k < n$ and does not depend on the irrelevant couplings λ_k with $k > n$. It is convenient to separate the λ_n -independent contributions, collected in δ_n, from those that are proportional to λ_n. All monomials $\prod_{k<n} \lambda_k^{n_k}$ contained in δ_n satisfy $\sum_{k<n} k n_k = n$, so they are at least quadratic in λ_k with $k < n$. Of course, β_{λ_n} can depend non-polynomially on the marginal couplings α.

An irrelevant deformation is made of a head and a queue. The head is the lowest-level irrelevant term. Denote its coupling with $\overline{\lambda}$. The queue is made of the other irrelevant terms, whose couplings λ_n are not independent, but unique functions of $\overline{\lambda}$ and α, given by certain "reduction relations" $\lambda_n = \lambda_n(\alpha, \overline{\lambda})$, to be determined. Differentiating the reduction relations with respect to the dynamical scale μ, the RG consistency conditions

$$\beta_{\lambda_n}(\alpha, \lambda) - \frac{\partial \lambda_n}{\partial \overline{\lambda}} \beta_{\overline{\lambda}} = \frac{\partial \lambda_n}{\partial \alpha} \beta_\alpha \tag{1.2}$$

are obtained. The consistency conditions (1.2) ensure that the divergences of the theory are renormalized just by the renormalization constants of α and $\overline{\lambda}$, plus field redefinitions. Nevertheless, (1.2) are not sufficient to determine the reduction relations uniquely, because their solutions contain arbitrary finite parameters ξ_n. Extra assumptions have to be introduced to have a true reduction.

In the realm of power-counting renormalizable theories similar problems were first considered by Zimmermann [4, 5], who suggested to eliminate the ξ-arbitrariness requiring that the reduction relations be analytic, for consistency with perturbation theory. The analytic reduction works in a set of models, when the reduced theory contains a single independent coupling, but is problematic when the reduced theory contains more than one independent coupling [6].

Zimmermann's approach can be understood as an alternative to unification. Its phenomenological implications have been investigated for example in [7]. For a technical review, see [8]. It is also possible to use Zimmermann's method to construct finite N=1 supersymmetric theories [9]. Beyond power-counting, Zimmermann's approach has been studied by Atance and Cortes in effective scalar theories and quantum gravity [10, 11] and by Kubo and M. Nunami [12] using the Wilsonian approach, but the systematics of the reduction of couplings in non-renormalizable theories (which I call *infinite reduction*) has not been developed, so far.

In the infinite reduction, some issues are different than in Zimmermann's reduction. First, note that no ξ-ambiguity affects the finite and quasi-finite non-renormalizable theories of ref.s [1, 2]. Indeed, the $\overline{\lambda}$-dependence of λ_n is unambiguously fixed on dimensional grounds: $\lambda_n = \overline{\lambda}^{n/\ell} f_n(\alpha)$, where ℓ is the level of $\overline{\lambda}$. So, when the renormalizable sector $\mathcal{R}$ is a conformal field theory $\mathcal{C}$ ($\beta_\alpha = 0$) the differential equations (1.2) collapse into algebraic equations. Because the equations are algebraic, the solutions do not contain new independent parameters. Because the equations are linear in their own unknowns λ_n, the solution exists and is unique, under certain conditions that are reviewed in sections 2 and 3. Finally, because δ_n depends only on the irrelevant couplings λ_m with $m < n$, the construction is algorithmic in the level n.

It was shown in [1, 2] that the free-field limit ($\alpha \to 0$) of the deformed theory is singular in α, and that the maximal singularity multiplying an irrelevant operator is bounded by the dimensionality of the operator itself, or, equivalently, by the power of $\overline{\lambda}$. These facts mean that: (i) the reduction is not analytic, but *meromorphic*; (ii) the singularity can be reabsorbed into $\overline{\lambda}$, defining a suitable "effective Planck mass" for the irrelevant interaction. A meromorphy of this type, where the negative powers can be arbitrarily high, but the maximal negative power grows linearly with the order of some expansion is called *perturbative meromorphy.*

Equipped with the knowledge learnt from ref.s [1, 2], I study prescriptions to remove the ξ-ambiguity in the irrelevant deformations of running renormalizable theories $\mathcal{R}$. I show that the reduction relations are uniquely determined by perturbative meromorphy around the free-field limit, if some existence conditions are fulfilled, e.g. certain linear combinations of one-loop anomalous dimensions, normalized with the one-loop coefficient of the $\mathcal{R}$ beta function, do not coincide with natural numbers. A non-trivial renormalization mixing makes the existence conditions less restrictive. Most of the ξ-arbitrariness is removed with this prescription, but sometimes the existence conditions are violated. Then, new independent couplings have to be introduced at high orders. In some models the number of independent couplings of the complete theory is finite, in other models it grows together with the order of the expansion. A form of predictivity is retained also in the latter case, because in general the growth is slow and a reasonably small number of parameters is sufficient to make predictions up to very high orders. Models of this type have been studied in [3].

The infinite reduction is scheme-independent, because the existence conditions involve only one-loop coefficients.

An alternative scheme-independent prescription for the infinite reduction is analyticity around an interacting fixed point of $\mathcal{R}$. In this case, the number of independent couplings generically remains finite in the complete reduced theory. Nevertheless, perturbative meromorphy around the free fixed point and analyticity around the interacting fixed point mutually exclude each other. Similarly, when $\mathcal{R}$ interpolates between two interacting fixed points, the reduction relations can be analytic only around one of them at a time. These features of the infinite reduction are a bit disappointing. However, it should be kept in mind that in the realm of non-renormalizable theories it is meaningful to impose conditions only around the IR fixed point, free or interacting, because the ultraviolet limit is not required to exist.

The study of quantum field theory beyond power counting has attracted interest for decades, motivated by low-energy QCD and quantum gravity. Some non-renormalizable models can be constructed with *ad hoc* procedures, such as the large N expansion, used for three-dimensional four-fermion theories [13] and sigma models [14]. Weinberg's asymptotic safety [15] is a more general scenario. The theory is assumed to have an interacting fixed point in the ultraviolet with a finite-dimensional critical surface. The RG flow tends to the fixed point only if the irrelevant couplings are appropriately fine-tuned. In general, only a finite number of arbitrary parameters survives this fine tuning. Asymptotic safety has been recently studied for gravity [16] and the Higgs sector of the Standard Model [17] using the ERG (exact renormalization-group) approach.

The paper is organized as follows. In sections 2 and 3 I review and elaborate on the finite and quasi-finite irrelevant deformations of interacting conformal field theories [1, 2]. In section 4 formulate the general principles of the infinite reduction and work out the conditions under which the number of independent parameters can be reduced and eventually kept finite. In sections 5 I propose an interpretation of the infinite reduction. In section 6 I study the infinite reduction around interacting fixed points. Section 7 contains some illustrative applications. In section 8 I discuss irrelevant deformations of theories that contain more than one marginal coupling. Section 9 contains the conclusions. Appendix A contains a brief review of Zimmermann's approach and a comparison with the infinite reduction. Appendix B contains definition and properties of perturbative meromorphy for the infinite reduction.

2. Finiteness beyond power-counting

Consider a conformal field theory $\mathcal{C}$ of fields φ interacting with the lagrangian $\mathcal{L}_{\mathcal{C}}[\varphi, \alpha]$, α denoting the marginal couplings. Let O_λ denote a basis of "essential", local, symmetric, scalar, canonically irrelevant operators constructed with the fields of $\mathcal{C}$ and their derivatives. The essential operators are the equivalence classes of operators that differ by total derivatives and terms proportional to the field equations [15]. Total derivatives are trivial in perturbation theory, while terms proportional to the field equations can be renormalized away by means of field redefinitions, so they do not affect the beta functions of the physical couplings. Finally, the operators O_λ are Lorentz scalars and have to be "symmetric", that is to say invariant under the non-anomalous symmetries of the theory, up to total derivatives.

The irrelevant terms can be ordered according to their level. If O_λ has canonical dimensionality d_λ in units of mass, then the level ℓ_λ of O_λ is the difference $d_\lambda - d$, d being the spacetime dimension. If λ denotes the coupling that multiplies the operator O_λ, then ℓ_λ is minus the naive dimensionality of λ. It is understood that in general each level contains finitely many operators, which can mix under renormalization. For the moment I do not distinguish operators of the same level. For concreteness, formulas are written in the case $d = 4$, because the generalization to other d's is simple.

The lagrangian of the deformed theory reads

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{C}}[\varphi, \alpha] + \sum_\lambda \lambda\ \mathcal{O}_\lambda(\varphi)\,.$$

The beta function β_λ of λ has the schematic structure (1.1) [1]

$$\beta_\lambda = \lambda\gamma_\lambda + \delta_\lambda\,. \tag{2.1}$$

obtained matching the naive dimensionalities, where δ_λ does not depend on λ and is polynomial, at least quadratic, in the irrelevant couplings λ' with levels $\ell_{\lambda'} < \ell_\lambda$. The coefficient

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$\gamma_\lambda(\alpha)$ of λ is the anomalous dimension of the operator O_λ, viewed as a composite operator of the undeformed theory $\mathcal{C}$. Operators with δ_λ equal to zero are called *protected*. Examples of protected operators are the chiral operators in four-dimensional supersymmetric theories [18]. Operators with $\gamma_\lambda = 0$ are finite, as viewed from the undeformed theory $\mathcal{C}$.

An irrelevant deformation is made of a *head* and a *queue*. The head is the first irrelevant term $\mathrm{O}_{\overline{\lambda}}$ of the deformation, multiplied by the independent coupling $\overline{\lambda}$. Obviously, $\delta_{\overline{\lambda}} = 0$, so the head is always protected. The queue is made of infinitely many irrelevant terms with levels $\ell_\lambda > \ell_{\overline{\lambda}}$, multiplied by unique functions of $\overline{\lambda}$ and α, obtained solving the finiteness equations

$$\beta_\lambda = 0\,. \tag{2.2}$$

Since δ_λ depends only on the couplings with levels $\ell_{\lambda'} < \ell_\lambda$ the solution can be worked out recursively in the levels ℓ_λ.

Equation (2.2) has solutions when the operator O_λ is not finite ($\gamma_\lambda \neq 0$) and when it is finite and protected ($\delta_\lambda = \gamma_\lambda = 0$). It does not have solutions when the operator O_λ is finite but not protected. The solution is trivial ($\lambda = 0$) when the operator is protected, but not finite.

The irrelevant deformation is non-trivial if the head $\mathrm{O}_{\overline{\lambda}}$ is a finite operator. Indeed, recalling that the head is always protected, the equation $\beta_{\overline{\lambda}} = 0$ leaves $\overline{\lambda}$ arbitrary. The queue exists if it does not include any finite unprotected operator, namely $\gamma_\lambda \neq 0$ any time $\delta_\lambda \neq 0$. When these *invertibility conditions* are fulfilled, the couplings of the queue are recursively given by

$$\lambda = -\frac{\delta_\lambda}{\gamma_\lambda} \tag{2.3}$$

in terms of $\overline{\lambda}$ and α.

The irrelevant deformation is trivial if the theory $\mathcal{C}$ has no finite irrelevant operator. Indeed, in this case $\beta_{\overline{\lambda}} = 0$ implies $\overline{\lambda} = 0$ and the other finiteness conditions iteratively imply that all λs vanish, which gives back the undeformed theory $\mathcal{C}$.

Summarizing, the theory $\mathcal{C}$ admits a non-trivial finite irrelevant deformation is there exists a finite operator and no finite unprotected operator.

The invertibility conditions are obviously violated if $\mathcal{C}$ is free, but are expected to be generically fulfilled if $\mathcal{C}$ is fully interacting. Examples of non-trivial finite unprotected irrelevant operators in interacting conformal field theories are not known. The known finite irrelevant operators, such as the chiral operators in four-dimensional superconformal field theories, are also protected.

The anomalous dimensions γ_λ depend on the marginal couplings α of $\mathcal{C}$. It is reasonable to expect that the anomalous dimension of an unprotected irrelevant operator vanishes at most for special values of α. In this sense, the requirement that the theory $\mathcal{C}$ does not have finite irrelevant unprotected operators is a restriction on the theory $\mathcal{C}$. Thus, in principle it is possible to say which conformal field theories admit which deformations just from the knowledge of the undeformed conformal theory, before turning the irrelevant deformation on.

When some invertibility conditions are violated, the irrelevant deformation cannot be finite, but can be easily promoted to a quasi-finite deformation (see the next section). More precisely, when an operator $\mathcal{O}_{\overline{\lambda}}$ of the queue is finite and unprotected (so the equation $\beta_{\overline{\lambda}} = 0$ has no solution) it is sufficient to treat the coupling $\overline{\lambda}$ as a new independent coupling, free to run according to the equation $\beta_{\overline{\lambda}} = \delta_{\overline{\lambda}}$. The rest of the queue is then determined as explained in the next section. These and other similar situations are discussed at length in the paper.

Thus, it is sufficient to assume that the invertibility conditions are violated in at most a finite number of cases to renormalize the irrelevant deformation with a finite number of independent couplings, plus field redefinitions.

Assuming that the invertibility conditions are fulfilled, the structure of the deformed lagrangian is

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{C}}[\varphi, \alpha] + \overline{\lambda}_\ell \mathcal{O}_\ell(\varphi) - \sum_{n=2}^{\infty} \frac{\delta_{n\ell}}{\gamma_{n\ell}} \, \mathcal{O}_{n\ell}(\varphi)\,. \tag{2.4}$$

Now, assume that $\mathcal{C}$ is finite, namely it belongs to a one-parameter family of conformal field theories that includes the free-field limit. In this case, $\beta_\alpha(\alpha) \equiv 0$ for every α. Then, the deformed theory (2.4) is renormalized just by field redefinitions, so it is finite. Instead, if $\mathcal{C}$ is the fixed point of an RG flow ($\beta_\alpha(\alpha_*) = 0$ only for some special value $\alpha = \alpha_*$), then "fake" renormalization constants are necessary to define both $\mathcal{C}$ and (2.4), including a non-trivial Z_α. Such renormalization constants do not cause the introduction of new physical couplings and do not affect the renormalization-group flow at $\alpha = \alpha_*$. It is natural to enlarge the notion of finiteness to include every theory of this type.

If the RG flow is defined varying the dynamical scale μ at fixed external momenta and $\overline{\lambda}_\ell$, then (2.4) is a fixed point of the flow. Instead, rescaling the overall energy E of correlations functions at fixed μ and $\overline{\lambda}_\ell$, every insertion of $\int \mathcal{O}_{n\ell}$ rescales, by construction, with canonical exponent $n\ell$. That means that the deformation (2.4) is irrelevant not only with respect to the gaussian fixed point, but also with respect to the interacting conformal field theory $\mathcal{C}$.

Operators of the same level can be distinguished with extra indices in λ, γ, δ and $\mathcal{O}$. The deformation then reads

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{C}}[\varphi, \alpha] + \overline{\lambda}_\ell^I \mathcal{O}_{\ell I}(\varphi) - \sum_{n=2}^{\infty} (\gamma_{n\ell}^{-1})^{IJ} \delta_{n,J} \, \mathcal{O}_{n\ell,I}(\varphi)\,, \tag{2.5}$$

where appropriate summations over $I, J \ldots$ are understood. By assumption, the matrix γ_ℓ^{IJ} should have a null eigenvector $\overline{\lambda}_\ell^I$. Then the operator $\overline{\lambda}_\ell^I \mathcal{O}_{\ell I}$ is finite and, used as the head of the queue, also protected, so the deformation is non-trivial. Instead, the matrices $\gamma_{n\ell}$, $n > 1$, should be invertible unless $\delta_n = 0$.

Assume that $\alpha = 0$ is the free-field limit of $\mathcal{C}$, and that α is defined so that the anomalous dimensions $\gamma_{n\ell}$ are generically of order α. In the presence of three-leg marginal vertices in four dimensions (multiplied by a coupling g such that $\alpha = g^2$) some non-diagonal entries of $\gamma_{n\ell}^{IJ}$ are of order g, due to the renormalization mixing. For the time being, I assume that there are no three-leg marginal vertices. The general case is treated in subsection 4.1.

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Some $\gamma_{n\ell}$ might have vanishing low-order coefficients. Call q_n the lowest non-vanishing order of $\gamma_{n\ell}$ and define $q = \max_n q_n$. When α is small, the $\lambda_{n\ell}$ behave at worst as

$$\lambda_{n\ell} \sim \frac{c_n \overline{\lambda}_\ell^n}{\alpha^{q(n-1)}}\,, \tag{2.6}$$

where c_n are constants. This result is proved by induction. It is certainly true for $n = 1$. Assume that it is true also for $\lambda_{k\ell}$, $k < n$. Since $\delta_{n\ell}$ depends on the λs of the lower levels $k\ell$, $k < n$, its behavior is at worst $\delta_{n\ell} \sim \prod_{k<n} \lambda_{k\ell}{}^{n_k}$, with $\sum_{k<n} k n_k = n$, where n_k are non-negative integers. Moreover, $m \equiv \sum_{k<n} n_k \geq 2$, since $\delta_{n\ell}$ is at least quadratic. Therefore

$$\lambda_{n\ell} = -\frac{\delta_{n\ell}}{\gamma_{n\ell}} \lesssim \frac{\overline{\lambda}_\ell^n}{\alpha^q} \prod_{k<n} \left(\frac{c_k}{\alpha^{q(k-1)}}\right)^{n_k} = \frac{c_n \overline{\lambda}_\ell^n}{\alpha^{q(n-m+1)}}\,, \tag{2.7}$$

which is at worst as singular as (2.6). Thus (2.6) is proved for arbitrary n. A behavior such as (2.6) is called *perturbatively meromorphic* of order q (see appendix B for more details).

Using this result, if $q < \infty$ the deformed lagrangian can be expressed as

$$\mathcal{L}[\varphi] \sim \mathcal{L}_{\mathcal{C}}[\varphi, \alpha] + \alpha^q \lambda_{\ell\mathrm{eff}} \mathcal{O}_\ell(\varphi) + \alpha^q \sum_{n=2}^{\infty} c_n(\alpha) \lambda_{\ell\mathrm{eff}}^n\ \mathcal{O}_{n\ell}(\varphi)\,, \tag{2.8}$$

where $\lambda_{\ell\mathrm{eff}} = \lambda_\ell \alpha^{-q}$ and the functions $c_n(\alpha)$ are analytic in α. The expansion in powers of the energy is meaningful for energies E much smaller than the “effective Planck mass” $M_{P\mathrm{eff}} \equiv 1/\lambda_{\ell\mathrm{eff}}^{1/\ell} = M_P \alpha^{q/\ell}$, where $M_P \equiv 1/\lambda_\ell^{1/\ell}$. The $\alpha \to 0$ limit at fixed $\lambda_{\ell\mathrm{eff}}$ is the gaussian fixed point, where the entire deformation disappears. On the other hand, the $\alpha \to 0$ limit at fixed λ_ℓ is singular. The effective Planck mass, and therefore the radius of convergence of the expansion, tend to zero when α tends to zero at fixed λ_ℓ. Thus the procedure cannot be used to construct irrelevant deformations of free-field theories.

In [1] it has been shown that three-dimensional quantum gravity coupled with interacting conformal matter can be quantized as a finite theory (see also [19]). This is due to the special properties of spacetime in three dimensions, because the Riemann tensor can be expressed in terms of the Ricci tensor and the scalar curvature. The theory is unique, and therefore predictive, because there is a unique finite protected operator, which is precisely the Einstein term. The results of the next sections can be used to generalize the construction of [1] and quantize also three-dimensional quantum gravity coupled with running matter (see section 7 for details).

In [2] finite chiral irrelevant deformations of superconformal field theories have been constructed in four dimensions. Such deformations are infinitely many, because all chiral operators are finite and protected.

3. Quasi finiteness beyond power-counting

When the conformal field theory $\mathcal{C}$ does not admit finite irrelevant operators, the finiteness equations have a trivial solution. Then it is natural to look for more general irrelevant deformations, relaxing the finiteness conditions in some way. A possibility is to demand that only a subset of beta functions vanish. In general, however, since the RG flow is a one-parameter flow, freezing one or some couplings freezes the entire flow to a point. This is consistent only if such a point is a fixed point of the flow, where all beta functions vanish. In less generic situations, when the set of couplings can be divided into two subsets g_i, and λ_j, such that the g-beta functions admit a factorization $\beta_g = h(g)f(g,\lambda)$, then it is meaningful to impose $\beta_g = 0$ at non-zero β_λ solving $h(g) = 0$. If $g_i = \overline{g}_i =$ constant denote the solutions of $h(g) = 0$, the RG flow of the couplings λ is non-trivial and consistently determined by the beta functions $\beta_\lambda(\overline{g}, \lambda)$.

Examples of this kind are the quasi-finite irrelevant deformations of interacting conformal field theories [2]. Again, the deformation is made of a head and a queue. The head is the irrelevant term with the lowest level, say ℓ, multiplied by the irrelevant coupling λ_ℓ, which is free to run. The queue runs coherently with the head and is made of terms of levels $n\ell$, with n integer, multiplied by unique functions of λ_ℓ and the marginal couplings α of $\mathcal{C}$. Since the irrelevant couplings $\lambda_{n\ell}$ are dimensionful, they can be conveniently split into an energy scale $1/\kappa$ and dimensionless ratios $r_n(\alpha)$:

$$\lambda_\ell = \kappa^\ell \,, \qquad \lambda_{n\ell} = r_n \; \lambda_\ell^n \,. \tag{3.1}$$

The structure of the beta functions β_{r_n} of the dimensionless couplings r_n are immediately derived from (2.1). Clearly, β_{r_n} cannot depend on κ, for dimensional reasons. Moreover, β_{r_n} is linear in r_n and at least quadratic in r_k with $k < n$, while the beta function of κ is proportional to κ:

$$\beta_{r_n}(r,\alpha) = \gamma_{r_n}(\alpha) r_n + d_n(r_<,\alpha) \,, \qquad \beta_\kappa = \frac{1}{\ell}\kappa\gamma_\ell(\alpha) \,, \tag{3.2}$$

where $d_n(r_<,\alpha)$ depends only on r_k with $k < n$ and $\gamma_{r_n}(\alpha) = \gamma_{n\ell}(\alpha) - n\gamma_\ell(\alpha)$. Then it is consistent to impose the *quasi-finiteness* conditions

$$\beta_{r_n}(r,\alpha) = 0 \,. \tag{3.3}$$

The solutions $\overline{r}(\alpha)$ exists if certain invertibility conditions, discussed below, hold. The lagrangian of the irrelevant deformation then reads

$$\mathcal{L}[\varphi] = \mathcal{L}_\mathcal{C}[\varphi,\alpha] + \kappa^\ell \mathcal{O}_\ell(\varphi) + \sum_{n=2}^{\infty} \overline{r}_n(\alpha) \; \kappa^{n\ell} \; \mathcal{O}_{n\ell}(\varphi) \,. \tag{3.4}$$

If $\mathcal{C}$ is finite, the deformed theory is renormalized by means of field redefinitions and a unique renormalization constant, the one for κ. The κ-running is determined by the RG equations:

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\ln\mu} = \beta_\kappa = \frac{\kappa}{\ell}\gamma_\ell(\alpha) \,, \qquad \kappa(\mu) = \kappa(\overline{\mu}) \left(\frac{\mu}{\overline{\mu}}\right)^{\gamma_\ell(\alpha)/\ell} \,.$$

For this reason, the theory (3.4) is called "quasi finite". It is natural to extend the notion of quasi finiteness to the irrelevant deformations of every (interacting) conformal field theory $\mathcal{C}$, in particular the fixed points of RG flows.

Finite and quasi-finite deformations differ for the κ-running and for the existence conditions (3.6) that I now discuss. Using (3.1) and (3.2), the equations (3.3) give immediately

$$r_n(\gamma_{n\ell} - n\gamma_\ell) = -d_n(r_<, \alpha)\,, \tag{3.5}$$

which can be solved iteratively in n if

$$\gamma_{n\ell} \neq n\gamma_\ell\,. \tag{3.6}$$

These invertibility conditions require that for every $n > 1$ the anomalous dimension of $\mathcal{O}_{n\ell}$ is not n times the anomalous dimension of the head. In the absence of symmetries or special protections, this is generically true. In any case, again, (3.6) is a restriction on the theory $\mathcal{C}$, so in principle it is possible to say which conformal theories admit which quasi-finite deformations before effectively turning the deformation on.

If some term $\mathcal{O}_{\ell'}$ of the queue violates (3.6) its coupling $\lambda_{\ell'}$ cannot run coherently with λ_ℓ and has to be treated as a new independent parameter. The resulting deformation is a multiple deformation, namely a deformation with more independent heads, of levels $\ell_1, \cdots \ell_k$, whose couplings λ_{ℓ_i} run independently. Formula (3.1) generalizes to

$$\lambda_m = \sum_{\{n\}} r_m^{n_1\cdots n_k}(\alpha)\ \lambda_{\ell_1}^{n_1}\cdots\lambda_{\ell_k}^{n_k}\,, \qquad \sum_{j=1}^{k} n_j\ell_j = m\,, \quad n_j \geq 0\,. \tag{3.7}$$

Here quasi-finiteness is the condition that the dimensionless coefficients $r_m^{n_1\cdots n_k}$ have vanishing beta functions. If the violations of the invertibility conditions are finitely many, then the deformation can be renormalized with a finite number of independent couplings, plus field redefinitions.

As in the previous section, assume that there are no three-leg vertices, that $\alpha = 0$ is the free-field limit of $\mathcal{C}$, and that α is defined so that the anomalous dimensions $\gamma_{n\ell}$ are generically of order α. When α is small, the behavior (2.6) and formula (2.8) hold also in the case of quasi-finite deformations. If $q < \infty$ the deformation is perturbatively meromorphic of order q in α and it is possible to define an effective Planck mass $M_{P\mathrm{eff}}$ such that the perturbative expansion in powers of the energy is meaningful for energies much smaller than $M_{P\mathrm{eff}}$. The gaussian fixed point is the $\alpha \to 0$ limit at fixed $M_{P\mathrm{eff}}$.

By construction, the deformation is irrelevant with respect to the gaussian fixed point and the weakly coupled conformal theories $\mathcal{C}$. For α large, if the invertibility conditions (3.6) are fulfilled, the deformation is relevant, marginal or irrelevant with respect to $\mathcal{C}$ if the head $\mathcal{O}_\ell$ is a relevant ($\gamma_\ell(\alpha) < -\ell$), marginal ($\gamma_\ell(\alpha) + \ell = 0$) or irrelevant ($\gamma_\ell(\alpha) > -\ell$) operator of $\mathcal{C}$, respectively. Indeed, rescale the overall energy E of correlations functions with respect to $1/\kappa$ and μ. Because of (3.1) and (3.3), the insertions of $\int \mathcal{O}_{n\ell}(\varphi)$ scale with exponents $n\ell + \beta_{n\ell}/\lambda_{n\ell} = n(\ell + \gamma_\ell)$.

4. Infinite reduction of couplings

In this section I study the infinite reduction for irrelevant deformations of running renormalizable theories $\mathcal{R}$. Again, I assume that $\mathcal{R}$ contains no relevant parameter. The formulas below are written, for simplicity, in the case that $\mathcal{R}$ contains just one marginal coupling α. The generalization to more marginal couplings is treated in section 8. As usual, divide the irrelevant deformation into levels, the level of an operator being its canonical dimensionality in units of mass minus the spacetime dimension d. For definiteness, I assume that d is four. The generalization to odd and other even dimensions is simple and left to the reader.

Let O_λ denote a basis of essential, local, symmetric, scalar, canonically irrelevant operators constructed with the fields of $\mathcal{R}$ and their derivatives. I assume that each level contains a finite number of terms. These in general mix under renormalization. To simplify the notation, I collectively denote the operators of level n with O_{λ_n}, $n = 1, 2, \ldots, \infty$, and their couplings with λ_n. When necessary, operators of the same level n can be distinguished with a second label $I = 1, \ldots N_n$ as shown in formula (2.4).

Again, the beta functions of the non-renormalizable theory

$$\mathcal{L}_{\mathrm{cl}}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, \alpha] + \sum_n \lambda_n \mathcal{O}_n(\varphi) \tag{4.1}$$

have the structure (2.1)

$$\beta_{\lambda_n}(\alpha, \lambda) = \gamma_n(\alpha)\, \lambda_n + \delta_n(\lambda_{m<n}, \alpha)\,, \tag{4.2}$$

where δ_n depends only on λ_p with $p < n$ and α, and is polynomial, at least quadratic, in the irrelevant couplings, while $\gamma_n(\alpha)$ is the anomalous dimension of O_{λ_n}, calculated in the undeformed theory $\mathcal{R}$.

As usual, the irrelevant deformation is made of a head $\mathcal{O}_\ell(\varphi)$, which is the irrelevant term with the lowest level, ℓ, and a queue. By dimensional arguments, the queue of the deformation is made only of terms whose levels are integer multiples of ℓ.

An *infinite reduction* of couplings is a set of functions

$$\lambda_{n\ell} = \lambda_{n\ell}(\lambda_\ell, \alpha) = f_n(\alpha)\lambda_\ell^n\,, \qquad n > 1\,, \tag{4.3}$$

subject to the conditions discussed below, such that the theory

$$\mathcal{L}_{\mathrm{cl}}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, \alpha] + \lambda_\ell \mathcal{O}_\ell(\varphi) + \sum_{n=2}^{\infty} f_n(\alpha)\lambda_\ell^n \mathcal{O}_{n\ell}(\varphi) \tag{4.4}$$

is renormalized by means of field redefinitions plus renormalization constants for α and λ_ℓ.

The beta functions (4.2) read

$$\beta_{n\ell} = \lambda_\ell^n \left[f_n(\alpha)\gamma_{n\ell}(\alpha) + \delta_n(f, \alpha) \right], \tag{4.5}$$

where $\delta_n(f, \alpha)$ depends polynomially, at least quadratically, on f_k with $k < n$, does not depend on f_k with $k \geq n$. Formulas (4.3) and (4.5) hold also for $n = 1$, if $f_1 = 1$ and $\delta_1 = 0$.

Differentiating the functions (4.3) with respect to the dynamical scale μ and using (4.5) the RG consistency equations

$$\left[\beta_\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} - \gamma_{n\ell}(\alpha) + n\gamma_\ell(\alpha)\right] f_n(\alpha) = \delta_n(f, \alpha) \tag{4.6}$$

are obtained.

Now I prove that (4.6) are necessary and sufficient conditions to renormalize the theory by means of renormalization constants just for λ_ℓ and α, plus field redefinitions. It is sufficient to focus the attention on the logarithmic divergences, setting the power-like divergences aside. Indeed, the power-like divergences are RG invariant and can be unambiguously subtracted away just as they come, without introducing new independent couplings. The logarithmic divergences can be studied at the level of the renormalization group, because the logarithms of the subtraction point μ are in one-to-one correspondence with the logarithms of the cut-off Λ.

Write the bare couplings $\lambda_{n\ell}(\Lambda)$ and $\alpha(\Lambda)$ in terms of their renormalization constants $Z_{n\ell}$ and Z_α in the minimal subtraction scheme,

$$\lambda_{n\ell}(\Lambda) = \lambda_{n\ell} Z_{n\ell}(\lambda, \alpha, \ln \Lambda/\mu)\,, \qquad \alpha(\Lambda) = \alpha Z_\alpha(\alpha, \ln \Lambda/\mu)\,,$$

$\lambda_{n\ell}$ and α being the renormalized couplings at the subtraction point μ. The renormalization of $\lambda_{n\ell}$ is not necessarily multiplicative (only the product $\lambda_{n\ell} Z_{n\ell}$ is analytic in $\lambda_{n\ell}$, for $n > 1$), but the compact notation $\lambda_{n\ell} Z_{n\ell}$ for $\lambda_{n\ell}(\Lambda)$ is convenient for the purposes of the infinite reduction. A more explicit notation is e.g. $\lambda_{n\ell}(\Lambda) = \lambda_{n\ell} + \Delta_{n\ell}(\lambda, \alpha, \ln \Lambda/\mu)$, with $\Delta_{n\ell}$ analytic in the couplings. The renormalization of λ_ℓ is obviously multiplicative.

Now, assume that the couplings $\lambda_{n\ell}$ are not independent, but satisfy (4.3) and (4.6). The RG consistency conditions (4.6) imply that the reduction relations have the same form at every energy scale, in particular at μ and Λ. Consequently,

$$\lambda_{n\ell} Z_{n\ell} = \lambda_{n\ell}(\Lambda) = f_n(\alpha(\Lambda))\lambda_\ell^n(\Lambda) = \lambda_\ell^n Z_\ell^n f_n(\alpha Z_\alpha)\,, \qquad n > 1\,. \tag{4.7}$$

This formula shows that the couplings $\lambda_{n\ell}$, $n > 1$, can be renormalized just attaching renormalization constants to λ_ℓ and α. The renormalization constants $Z_{n\ell}$, $n > 1$, are not independent. Indeed, (4.7) implies

$$Z_{n\ell} = Z_\ell^n \frac{f_n(\alpha Z_\alpha)}{f_n(\alpha)}\,.$$

Despite these facts, no true reduction of couplings is achieved simply solving the RG consistency conditions (4.6). Indeed, (4.6) are differential equations for the unknown functions $f_n(\alpha)$, $n > 1$. The solutions depend on arbitrary constants ξ. From the point of view of renormalization, the arbitrary constants ξ are finite parameters, namely $Z_\xi \equiv 1$. The equations (4.6) and the arguments leading to (4.7) are simply a rearrangement of the renormalization of the theory, with no true gain, because the number of renormalization constants is reduced at the price of introducing new functions f_n. To remove the ξ-ambiguities contained in the solutions of (4.6) and achieve a true reduction of couplings,

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extra assumptions have to be made. Guided by the experience of finite and quasi-finite theories, it is natural to remove the ξ-arbitrariness requiring that the solution $f_n(\alpha)$ be meromorphic in α.

For the moment I assume that $\beta_\alpha^{(1)} \neq 0$ and that $\mathcal{R}$ contains only four-leg marginal vertices, i.e. it is the φ^4 theory in four dimensions (but similar arguments apply if $\mathcal{R}$ the φ^6 theory in three dimensions). The marginal coupling α is defined so that the beta function and the anomalous dimensions of $\mathcal{R}$ have expansions

$$\beta_\alpha = \alpha^2\beta_\alpha^{(1)} + \mathcal{O}(\alpha^3), \qquad \gamma_n(\alpha) = \alpha\gamma_n^{(1)} + \mathcal{O}(\alpha^2), \tag{4.8}$$

etc. The models with marginal three-leg vertices (multiplied by g such that $\alpha = g^2$) are treated in subsection 4.1. I prove that if the invertibility conditions

$$r_{n,\ell} \equiv \frac{1}{\beta_\alpha^{(1)}}\left(\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)}\right) + n - 1 \notin \mathbb{N}, \qquad n > 1, \tag{4.9}$$

are fulfilled, there exist unique meromorphic reduction relations of the form

$$\lambda_{n\ell} = f_n(\alpha)\lambda_\ell^n = \frac{\lambda_\ell^n}{\alpha^{n-1}}\sum_{k=0}^{\infty} d_{n,k}\alpha^k, \tag{4.10}$$

with unambiguous numerical coefficients $d_{n,k}$.

The result is proved by induction. Clearly, (4.10) is true for $n = 1$. Assume that $\lambda_{j\ell}$ with $j < n$ satisfy (4.10). By the usual dimensional arguments $\delta_{n\ell} \sim \prod_{j<n} \lambda_{j\ell}{}^{n_j}(1+\mathcal{O}(\alpha))$, with $\sum_{j<n} jn_j = n$, where n_j are non-negative integers, and $m \equiv \sum_{j<n} n_j \geq 2$, since $\delta_{n\ell}$ is at least quadratic. Therefore for small α the inductive hypothesis implies

$$\delta_{n\ell} \sim \lambda_\ell^n \prod_{j<n}\left(\frac{1}{\alpha^{j-1}}\right)^{n_j} = \frac{\lambda_\ell^n}{\alpha^{n-m}} \leq \frac{\lambda_\ell^n}{\alpha^{n-2}}.$$

Now, insert the ansatz (4.10) into the RG consistency conditions (4.6) and solve for $d_{n,k}$ recursively in k. If the invertibility conditions (4.9) hold, the solution is well defined and the coefficients $d_{n,k}$ have unambiguous values of the form

$$d_{n,k} = \frac{P_{n,k}}{\prod_{j=1}^{k+1}\left(\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)} + (n-j)\beta_\alpha^{(1)}\right)}, \tag{4.11}$$

where $P_{n,k}$ depends polynomially on the coefficients of the beta function and the anomalous dimensions and on $d_{m,k}$ with $m < n$. This proves the statement for arbitrary n. In general the numerator in (4.11) does not vanish when the denominator vanishes.

Formula (4.10) shows that the irrelevant deformation (4.4) is perturbatively meromorphic of order one. Defining $\lambda_{\ell\text{eff}} = \lambda_\ell/\alpha = 1/M_{P\text{eff}}^\ell$, the behavior of the lagrangian for small α is

$$\mathcal{L}[\varphi] \sim \mathcal{L}_\mathcal{R}[\varphi,\alpha] + \alpha\lambda_{\ell\text{eff}}\mathcal{O}_\ell(\varphi) + \alpha\sum_{n=2}^{\infty} d_{n,0}\lambda_{\ell\text{eff}}^n\ \mathcal{O}_{n\ell}(\varphi). \tag{4.12}$$

The perturbative expansion is meaningful for $\alpha \ll 1$, for energies $E \ll M_{P\text{eff}}$. Therefore, at fixed $\lambda_{\ell\text{eff}}(\mu)$ the irrelevant deformation disappears in the limit where the renormalizable sector becomes free.

Clearly, the invertibility conditions (4.9) are sufficient to have a meaningful reduction that is perturbatively meromorphic of order one. The conditions (4.9) are not necessary, because in some cases δ_n might start from higher orders in α, and f_n be less singular than (4.10). Formula (4.10) just describes the worst behavior. Observe that the quantities $r_{n,\ell}$ depend only on one-loop coefficients, yet they determine the existence of the reduction to all orders. Moreover, the $r_{n,\ell}$s are just rational numbers and it is not unfrequent that they coincide with natural numbers for some ns. The violations of the invertibility conditions can be cured introducing new independent couplings (see below).

The parametrization (4.12) in terms of α and $\lambda_{\ell\text{eff}}$ is non-minimal, in the sense that no irrelevant vertex is multiplied by $\lambda_{\ell\text{eff}}$, all irrelevant vertices being multiplied by products $\alpha\lambda_{\ell\text{eff}}^n$. An example of minimal parametrization of the space of couplings is (4.4), where the head is multiplied just by λ_ℓ. All minimal parametrizations are singular for $\alpha \to 0$. The reason is that marginal and irrelevant deformations do not commute. In particular, it is necessary to have $\alpha \neq 0$ to build the irrelevant deformation, but the marginal interaction exists also in the absence of irrelevant deformations.

4.1 Three-leg marginal vertices and renormalization mixing

Taking care of the renormalization mixing, in the absence of three-leg marginal vertices the invertibility conditions become a straightforward matrix generalization of (4.9) [3]. Instead, when $\mathcal{R}$ contains three-leg marginal vertices, multiplied by a coupling g such that $\alpha = g^2$ (e.g. $\mathcal{R}$ is a gauge theory), the effects of the renormalization mixing are non-trivial. It is convenient to define a parity transformation, called U, that sends g into $-g$ and every field φ into $-\varphi$. Clearly, $\mathcal{R}$ is U -invariant. Assigning suitable U-parities to the irrelevant couplings λ_n also (4.1) is U-invariant. Observe that $\delta_n(\lambda_{m<n}, \alpha)$ can contain non-negative powers of both α and g. To simplify the treatment, it is convenient to work with U-even quantities whenever possible, which can be achieved with a simple trick. Define $\widehat{\mathcal{O}}_{n\ell}(\varphi) \equiv g^{N_{n\ell}-2}\mathcal{O}_{n\ell}(\varphi)$ and $\widehat{\lambda}_{n\ell}$ such that $\widehat{\lambda}_{n\ell}\widehat{\mathcal{O}}_{n\ell}(\varphi) = \lambda_{n\ell}\mathcal{O}_{n\ell}(\varphi)$, where $N_{n\ell}$ is the number of legs of the vertex $\mathcal{O}_{n\ell}(\varphi)$. Generalizing (4.3) to

$$\widehat{\lambda}_{n\ell} = \widehat{\lambda}_{n\ell}(\widehat{\lambda}_\ell, \alpha) = \widehat{f}_n(\alpha)\widehat{\lambda}_\ell^n\,, \qquad n > 1\,, \tag{4.13}$$

it is clear that every $\widehat{f}_n$ is even. Simple diagrammatics show that $\delta_{n\ell}(\lambda, \alpha) = g^{N_{n\ell}}\widehat{\delta}_{n\ell}(\widehat{\lambda}, \alpha)$, where $\widehat{\delta}_{n\ell}(\widehat{\lambda}, \alpha)$ is polynomial in the $\widehat{\lambda}_{k\ell}$'s, $k < n$, and analytic in α and does not depend on the $\widehat{\lambda}_{k\ell}$'s with $k > n$. Indeed, let G be a diagram contributing to $\delta_{n\ell}$, with E external legs, I internal legs and V vertices. The g-powers carried by the vertices are equal to $E + 2I$, which is the number of legs carried by the vertices, minus $2V$. Since $I - V = L - 1 \geq 0$, where L is the number of loops, and $N_{n\ell} = E$, the result follows. By the same argument, the anomalous dimensions $\widehat{\gamma}_{n\ell}$ of the operators $\widehat{\mathcal{O}}_{n\ell}(\varphi)$ are analytic in α and at one loop they are of order α. The reduction equations for the $\widehat{f}_n$'s are then

$$\left[\beta_\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} - \widehat{\gamma}_{n\ell}(\alpha) + n\widehat{\gamma}_\ell(\alpha)\right] \widehat{f}_n(\alpha) = \alpha\widehat{\delta}_{n\ell}(\widehat{f}, \alpha)\,. \tag{4.14}$$

If the invertibility contitions

$$\frac{1}{\beta_\alpha^{(1)}}\left(\widehat{\gamma}_{n\ell}^{(1)}-n\widehat{\gamma}_\ell^{(1)}\right)\notin\mathbb{N}\,,\qquad n>1\,,\tag{4.15}$$

hold, the equations (4.14) admit unique solutions $\widehat{f}_n(\alpha)$ analytic in α. In this parametrization, $\widehat{\lambda}_\ell$ coincides with $\lambda_{\ell\mathrm{eff}}$. Since each irrelevant vertex has at least three legs, the deformation

$$\mathcal{L}[\varphi]\sim\mathcal{L}_{\mathcal{R}}[\varphi,\alpha]+g^{N_\ell-2}\lambda_{\ell\mathrm{eff}}\mathcal{O}_\ell(\varphi)+\sum_{n=2}^{\infty}g^{N_{n\ell}-2}\widehat{f}_n(\alpha)\lambda_{\ell\mathrm{eff}}^n\ \mathcal{O}_{n\ell}(\varphi)\tag{4.16}$$

is perturbatively meromorphic of order g, instead of α.

Now, consider the renormalization mixing, calculated in the undeformed theory $\mathcal{R}$, among operators with the same dimensionality $n\ell$ in units of mass, $n\geq 1$. Distinguish the mixing operators with indices $I,J,\ldots$. If ℓ is the level of the deformation, denote the inequivalent operators of level ℓ with $\widehat{\mathcal{O}}_\ell^I$, the coefficient-matrix of their one-loop anomalous dimensions with $\widehat{\gamma}_\ell^{(1)\ IJ}$, an eigenvalue of $\widehat{\gamma}_\ell^{(1)\ IJ}$ with $\gamma_\ell^{(1)}$ and the corresponding eigenvector with d_0^I. For simplicity, assume that $\gamma_\ell^{(1)}$ is real. Below I describe how the arguments are modified when $\gamma_\ell^{(1)}$ is complex. Denote the operators of the queue of the deformation with $\widehat{\mathcal{O}}_{n\ell}^I$, $n>1$, and their couplings with $\widehat{\lambda}_{n\ell}^I$. In the hatted notation introduced above the matrices of anomalous dimensions $\widehat{\gamma}_{n\ell}^{IJ}$ are analytic in α and at one loop they are of order α. The hatted beta functions read

$$\widehat{\beta}_{n\ell}^I=\sum_J\widehat{\gamma}_{n\ell}^{IJ}(\alpha)\widehat{\lambda}_{n\ell}^J+\alpha\widehat{\delta}_{n\ell}^I\,,$$

where $\widehat{\delta}_{n\ell}^I$ depends only on $\widehat{\lambda}_{m\ell}^I$ with $m<n$ and is analytic in α. Introduce an auxiliary coupling $\widehat{\lambda}_\ell$ of level ℓ, with beta function $\widehat{\beta}_{\lambda_\ell}=\gamma_\ell^{(1)}\alpha\widehat{\lambda}_\ell$. The beta function of $\widehat{\lambda}_\ell$ can be chosen to be one-loop exact with an approprite scheme choice (any other choice being equivalent to a redefinition $\widehat{\lambda}_\ell\rightarrow h(\alpha)\widehat{\lambda}_\ell$, with $h(\alpha)$ analytic in α, $h(0)=1$). The reduction relations have the form

$$\widehat{\lambda}_{n\ell}^I=\widehat{f}_n^I(\alpha)\widehat{\lambda}_\ell^n\,,\qquad n\geq 1\,,$$

where $\widehat{f}_n^I(\alpha)$ are analytic in α. If k is a natural number, it is straightforward to check that the existence conditions are that the matrices

$$\widehat{r}_{n,k,\ell}^{IJ}=\widehat{\gamma}_{n\ell}^{(1)\ IJ}-n\gamma_\ell^{(1)}\delta^{IJ}-k\beta_\alpha^{(1)}\delta^{IJ}\,,\tag{4.17}$$

be invertible for $n>1$, $k\geq 0$ and for $n=1$, $k>0$. If the invertibility conditions are fulfilled, the solution is uniquely determined in terms of d_0^I. The head of the deformation is $\sum_I\widehat{\mathcal{O}}_\ell^I\widehat{\lambda}_\ell^I$.

The entries of the matrices $\widehat{\gamma}_{n\ell}^{(1)\ IJ}$ are rational numbers divided by $\pi^{d/2}$. For the purposes of the infinite reduction, the renormalization mixing is non-trivial when the matrix $\widehat{\gamma}_{n\ell}^{(1)\ IJ}$ is non-triangular. In general, the size of the non-triangular blocks of $\widehat{r}_{n,k,\ell}^{IJ}$ grows with n. A renormalization mixing with these properties makes the violations of the existence conditions much rarer, since the eigenvalues of a non-triangular matrix with rational entries are in general irrational or complex. Below I explain that any time the invertibility conditions are violated a new coupling has to be introduced. It is reasonable to expect that a sufficiently non-trivial renormalization mixing causes at most the sporadic appearance of a finite number of new couplings.

Multiple-head deformations are treated as explained at the end of section 3, see formula (3.7). If the eigenvalue $\gamma_\ell^{(1)}$ is complex it is necessary to consider its complex conjugate $\overline{\gamma}_\ell^{(1)}$ and the corresponding eigenvector $\overline{d}_0^I$ together with $\gamma_\ell^{(1)}$ and d_0^I, and introduce the conjugate auxiliary coupling $\widehat{\overline{\lambda}}_\ell$, with beta function $\widehat{\beta}_{\overline{\lambda}_\ell}=\overline{\gamma}_\ell^{(1)}\alpha\widehat{\overline{\lambda}}_\ell$. The reduction relations are expansions of the form (3.7) in powers of $\widehat{\lambda}_\ell$ and $\widehat{\overline{\lambda}}_\ell$, and have to satisfy straightforward reality conditions. This gives in practice a two-head deformation. Alternatively, it is possible to use a real two-by-two matrix (the real Jordan canonical form) in place of $\gamma_\ell^{(1)}$ and then proceed as for two-head deformations, without the need of reality conditions.

In each model, the more appropriate invertibility conditions are (4.15) or (4.9) depending on the presence or absence of marginal three-leg vertices. Perturbative meromorphy is described by (4.16) or (4.12), respectively. To keep the notation to a minimum, in the rest of the paper I work in the absence of marginal three-leg vertices, since it is straightforward to adapt the arguments to the other case when necessary. More details can be found in [20].

4.2 Dependence on the arbitrary constants and uniqueness of the infinite reduction

When the invertibility conditions (4.9) are fulfilled, the most general solution of (4.6) is

$$\frac{1}{\alpha^{n-1}}\sum_{k=0}^{\infty}d_{n,k}\alpha^k+\xi_n\overline{s}_n(\alpha)\,, \tag{4.18}$$

where $d_{n,k}$ are given in (4.11), ξ_n is an arbitrary constant and the function

$$\overline{s}_n(\alpha)=\exp\left(\int^{\alpha}\mathrm{d}\alpha'\frac{\gamma_{n\ell}(\alpha')-n\gamma_\ell(\alpha')}{\beta_\alpha(\alpha')}\right) \tag{4.19}$$

is the solution of the homogeneous equation. For α small

$$\overline{s}_n(\alpha)\sim\alpha^{Q_n}\,,\qquad Q_n=\frac{\gamma_{n\ell}^{(1)}-n\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}}\,.$$

Formula (4.18) can be used also to study the solutions of (4.6) when the invertibility conditions are violated, with a suitable process of limit. Three situations can take place:

(i) If Q_n is not integer $\overline{s}_n(\alpha)$ is not meromorphic. In this case the invertibility conditions are fulfilled and perturbative meromorphy fixes $\xi_n=0$, thereby selecting the reduction uniquely.

(ii) When Q_n is an integer $\overline{p}\geq -n+1$ the invertibility conditions (4.9) are violated at order $\overline{p}+n-1$. To study this case, use (4.18) to approach the case $Q_n=\overline{p}$ from

$Q_n = \overline{p} + \delta$, δ irrational, taking the limit $\delta \to 0$. The singularity $\sim 1/\delta$ in $d_{n,\overline{p}+n-1}$ can be removed redefining the constant ξ_n,

$$\frac{1}{\alpha^{n-1}} \frac{b_{n,\overline{p}+n-1}\alpha^{\overline{p}+n-1}}{Q_n - \overline{p}} + \xi_n \alpha^{Q_n} = \alpha^{\overline{p}} \left(\frac{b_n}{\delta} + \xi_n \alpha^{\delta} \right) \sim \alpha^{\overline{p}} \left(-b_n \ln \alpha + \xi_n' \right). \quad (4.20)$$

Here $b_{n,\overline{p}+n-1}$ is a known non-singular factor. Formula (4.20) shows that meromorphy is violated by a logarithm and no value of ξ_n' can eliminate it. The violation of meromorphy can be reabsorbed only introducing a new independent coupling (see below), which reabsorbs also the singularities of $d_{n,k}$ with $k > \overline{p} + n - 1$. The difference between this case and case (i) is that here the introduction of the new coupling is compulsory, while in case (i) the violation of meromorphy can be removed with an appropriate choice of the arbitrary constant ξ_n.

(iii) When Q_n is an integer $< -n+1$ the invertibility conditions (4.9) are fulfilled and the solution (4.18) is meromorphic for arbitrary ξ_n. However, the order of perturbative meromorphy increases.

Now I study cases (ii) and (iii) in more detail.

4.3 Case (ii): violations of the invertibility conditions and introduction of new parameters at higher orders

Suppose that $r_{\overline{n},\ell}$ is a natural number $\overline{k}$ for some $\overline{n}$ or that some matrix $r^{IJ}_{n,k,\ell}$ has a null eigenvector. Then the reduction fails at the $\overline{k}$th order, unless a new independent parameter $\overline{\lambda}_{\overline{n}\ell}$ is introduced at that order in front of $\mathcal{O}_{\overline{n}\ell}$. Write

$$\lambda_{\overline{n}\ell} = \frac{1}{\alpha^{\overline{n}-1}} \left[\lambda_\ell^{\overline{n}} \sum_{j=0}^{\overline{k}-1} d_{\overline{n},j} \alpha^j + \alpha^{\overline{k}} \overline{\lambda}_{\overline{n}\ell} \right], \quad (4.21)$$

where $d_{\overline{n},j}$, $j < \overline{k}$ are calculated as above. The new parameter $\overline{\lambda}_{\overline{n}\ell}$ hides the logarithm of (4.20). Its beta function has the form

$$\overline{\beta}_{\overline{n}\ell} = \overline{\gamma}_{\overline{n}\ell}(\alpha)\, \overline{\lambda}_{\overline{n}\ell} + \overline{\delta}_{\overline{n}\ell}(\lambda_{m<\overline{n}}, \alpha), \qquad \overline{\gamma}_{\overline{n}\ell}(\alpha) = \overline{n}\gamma_\ell^{(1)}\alpha + \mathcal{O}(\alpha^2), \qquad \overline{\delta}_{\overline{n}\ell} = \lambda_\ell^{\overline{n}} \mathcal{O}(\alpha),$$

The one-loop coefficient of $\overline{\gamma}_{\overline{n}\ell}$ is derived from the equality $r_{\overline{n},\ell} = \overline{k}$.

The introduction of $\overline{\lambda}_{\overline{n}\ell}$ affects also the reduction relations for $n > \overline{n}$. Observing that the $\overline{\lambda}_{\overline{n}\ell}$ contributes only from order $\overline{k}$, the modified reduction relations for $n > \overline{n}$ read

$$\lambda_{n\ell} = \frac{1}{\alpha^{n-1}} \sum_{q=0}^{[n/\overline{n}]} \alpha^{\overline{k}q} a_{n\ell}^{(q)}(\alpha) \lambda_\ell^{n-\overline{n}q} \overline{\lambda}_{\overline{n}\ell}^q, \qquad n > \overline{n}, \quad (4.22)$$

where [ ] denotes the integral part and the coefficients $a_{n\ell}^{(q)}$ are power series in α. Inserting (4.22) in (4.6) the coefficients $a_{n\ell}^{(q)}$ are worked out iteratively from $q = [n/\overline{n}]$ to $q = 0$, term-by-term in the α-expansion. The existence conditions for $a_{nm}^{(q)}$ are

$$r_{n,\ell,q} = r_{n,\ell} - \overline{k}q \notin \mathbb{N} \quad (4.23)$$

and do not add further restrictions, because they are already contained in (4.9).

When a further invertibility condition (4.9), $n > \overline{n}$, is violated, the story repeats. A new parameter $\overline{\lambda}_{n\ell}$ is introduced at order $\alpha^{r_{n,\ell}}$. If several conditions (4.23), for different values of q, are violated at the same time, all singular monomials of (4.22) are reabsorbed into the same new parameter $\overline{\lambda}_{n\ell}$. Observe that the reduction itself guides the introduction of the new parameters.

Due to (4.21) and (4.22), after the introduction of the new parameters $\overline{\lambda}_{\overline{n}\ell}$ the deformation is still perturbatively meromorphic of order one: it is sufficient to define $\overline{\lambda}_{\overline{n}\ell\mathrm{eff}} = \alpha^{-\overline{n}}\overline{\lambda}_{\overline{n}\ell}$ and take α small at fixed $M_{P\mathrm{eff}}$ anf $\overline{\lambda}_{\overline{n}\ell\mathrm{eff}}$.

Violations of the invertibility conditions (4.9) or (4.17), although unfrequent, can occur. In some models the renormalization mixing can be sufficiently non-trivial to keep the violations to a finite number. Then the final theory is predictive, in the sense that it is renormalized with a finite number of independent couplings and renormalization constants, plus field redefinitions. It is possible to have a form of predictivity also when the violations of the invertibility conditions are infinitely many. Indeed, because of formula (4.9), it is reasonable to expect that in general the quantity $r_{n,\ell}$ grows with n. This ensures that the new parameters $\overline{\lambda}_{\overline{n}\ell}$ are sporadically introduced at increasingly high orders in α. Even if the final theory contains infinitely independent couplings, a finite subset of them is sufficient to make high-order predictions. In ref. [3] I have given models that illustrate these facts. Consider the scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 + V(\varphi, \partial\varphi, \partial^2\varphi, \ldots) = \frac{1}{2}(\partial\varphi)^2 + {\sum_{n=0}^{\infty}}' V_n(\varphi)[\partial^{2n}\varphi]\,, \tag{4.24}$$

where $[\partial^{2n}\varphi]$ is a compact notation to denote $2n$ variously distributed derivatives of the field φ, contracted in all possible ways, and the primed sum runs over a basis of terms that are inequivalent with respect to field redefinitions and additions of total derivatives.

Assume that the head of the irrelevant deformation is the operator $\varphi^{\ell+4}$. At one and two loops the potential $V_0(\varphi)$ does not mix with derivative terms. This ensures that the invertibility conditions for the monomials $\varphi^{n\ell+4}$ have the form $r_{n,\ell} \notin \mathbb{N}$ with $r_{n,\ell}$ given by (4.9). Inserting the one-loop values of the anomalous dimensions and beta functions, the invertibility conditions are

$$r_{n,\ell} = \frac{1}{6}(n-1)(n\ell^2 - 6) \notin \mathbb{N} \tag{4.25}$$

for $n > 1$. The condition is violated in infinitely many cases. When $r_{\overline{n},\ell} = \overline{k}(\overline{n}) \in \mathbb{N}$ for some $\overline{n}$, a new parameter is introduced at order $\overline{k}(\overline{n})$. For example, for $\ell = 2$, which is the theory $\varphi^4 + \varphi^6$, the first integer values of $r_{n,2}$ are 2,5,15,22,40. . ., so the first new parameter appears at two loops. The terms with $n > 0$ in (4.24) provide other invertibility conditions, similar to (4.25), and from a certain point onwards the renormalization mixing becomes non-trivial. So, formula (4.25) is sufficient to estimate the growth of the number of parameters and ensures that it is possible to make calculations up to forty loops using about ten independent couplings.

4.4 Case (iii): properties of the effective Planck mass

If the invertibility conditions are fulfilled, but $r_{\overline{n},\ell} = -\overline{r}$ is a negative integer for some $\overline{n}$ then (4.18) and (4.19) show that the solution $f_{\overline{n}}(\alpha)$ admits an arbitrary parameter $\overline{d}$ multiplying a more singular meromorphic expansion

$$f_{\overline{n}}(\alpha) = \frac{\overline{d}}{\alpha^{\overline{n}-1+\overline{r}}} \sum_{k=0}^{\infty} c_{\overline{n},k}\alpha^k + \frac{1}{\alpha^{\overline{n}-1}} \sum_{k=0}^{\infty} d_{\overline{n},k}\alpha^k . \tag{4.26}$$

The coefficients $c_{\overline{n},k}$ and $d_{\overline{n},k}$ are uniquely determined, with $c_{\overline{n},0} = 1$. I assume that for $n < \overline{n}$ the functions f_n behave as in (4.10). By theorem 3 of appendix B, if some more restrictive invertibility conditions are fulfilled for $n > \overline{n}$ (see formula (4.29) below), the behavior of $f_n(\alpha)$ for arbitrary n is

$$f_n(\alpha) \sim \frac{1}{\alpha^{n-1+\overline{r}[n/\overline{n}]}}$$

and the irrelevant deformation with heads λ_ℓ and $\widehat{\lambda}_{\overline{n}\ell} \equiv \overline{d}\lambda_\ell^{\overline{n}}$ is perturbatively meromorphic of order

$$\overline{q} = 1 + \left[\frac{\overline{r}}{\overline{n}-1}\right]_+ , \tag{4.27}$$

$[x]_+$ denoting the minimum integer $\geq x$. The effective Planck mass is defined by $\lambda_{\ell\text{eff}} = \lambda_\ell\alpha^{-\overline{q}} = 1/M_{P\text{eff}}^\ell$ and $\widehat{\lambda}_{\overline{n}\ell\text{eff}} = \alpha^{-\overline{n}\overline{q}}\widehat{\lambda}_{\overline{n}\ell} = \overline{d}/M_{P\text{eff}}^{\overline{n}\ell}$: for $\alpha \sim 0$

$$\mathcal{L}[\varphi] \sim \mathcal{L}_{\mathcal{R}}[\varphi,\alpha] + \alpha^{\overline{q}}\lambda_{\ell\text{eff}}\mathcal{O}_\ell(\varphi) + \alpha^{\overline{q}} \sum_{n=2}^{\infty} a_n\alpha^{p_n}\lambda_{\ell\text{eff}}^n\ \mathcal{O}_{n\ell}(\varphi) , \tag{4.28}$$

where a_n are factors that depend also on the arbitrary parameter $\overline{d}$ and p_n are non-negative integers. From (4.28) it follows that sufficient invertibility conditions for $n > \overline{n}$ are

$$r'_{n,\ell} \equiv \frac{1}{\beta_\alpha^{(1)}}\left(\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)}\right) + \overline{q}(n-1) \notin \mathbb{N} , \qquad n > \overline{n} . \tag{4.29}$$

If some other $r'_{n,\ell}$, $n > \overline{n}$, is an integer, the procedures described so far can be iterated straightforwardly.

In the case just studied the meromorphic reduction admits arbitrary finite parameters $\overline{d}$. When a new coupling of type $\widehat{\lambda}_{\overline{n}\ell}$ is introduced, the order $\overline{q}$ of perturbative meromorphy increases. The effective Planck mass $M_{P\text{eff}} = M_P\alpha^{\overline{q}/\ell}$ becomes smaller for $\alpha \to 0$ at fixed M_P. Equivalently, the irrelevant interaction becomes weaker for $\alpha \to 0$ at fixed $M_{P\text{eff}}$, as shown by formula (2.8). Since the expansion in powers of the energy is meaningful for $E \ll M_{P\text{eff}}$, a smaller effective Planck mass means a more restricted perturbative domain. It is meaningful to require that the perturbative domain be maximal, in which case all extra finite parameters $\overline{d}$ have to be switched off.

4.5 $\mathcal{R}$-beta function with some vanishing coefficients

So far I have assumed $\beta_\alpha^{(1)} \neq 0$. When $\beta_\alpha^{(1)} = 0$ the invertibility conditions (4.9) for the existence of the coefficients $d_{n,k}$ in (4.11) simplify and become just one for every n, namely

$$\gamma_{n\ell}^{(1)} \neq n\gamma_\ell^{(1)} \,. \tag{4.30}$$

This situation is an interesting generalization of the finite and quasi-finite theories of sections 2 and 3: when $\beta_\alpha \neq 0$, but $\beta_\alpha^{(1)} = 0$ the conditions (3.6) are replaced by their one-loop counterparts (4.30). Here $\overline{s}_n(\alpha)$ has an essential singularity, so perturbative meromorphy implies $\xi_n = 0$ for every n.

When $\beta_\alpha^{(1)} = 0$ and (4.30) are fulfilled for $n \neq \overline{n}$, but violated for $n = \overline{n}$, the invertibility conditions for $f_{\overline{n}}(\alpha)$ are

$$r_{\overline{n},\ell}^{(2)} = Q_{\overline{n}}^{(2)} + \overline{n} \equiv \frac{1}{\beta_\alpha^{(2)}}\left(\gamma_{\overline{n}\ell}^{(2)} - \overline{n}\gamma_\ell^{(2)}\right) + \overline{n} \notin \mathbb{N} \,, \tag{4.31}$$

assuming $\beta_\alpha^{(2)} \neq 0$. Then $f_{\overline{n}}(\alpha)$ has a more singular expansion

$$f_{\overline{n}}(\alpha) = \frac{1}{\alpha^{\overline{n}}} \sum_{k=0}^{\infty} d_{\overline{n},k-1}\alpha^k \,.$$

The invertibility conditions for $n > \overline{n}$ are not modified. Applying theorem 3 of appendix B with $q = 1$ and $\overline{r} = 1$ it follows that when the unmodified existence conditions for $n > \overline{n}$ are fulfilled, (4.10) is replaced by the more singular expansion

$$f_n(\alpha) = \frac{1}{\alpha^{n-1+[n/\overline{n}]}} \sum_{k=0}^{\infty} d_{n,k-[n/\overline{n}]}\alpha^k \,. \tag{4.32}$$

The coefficients $d_{n,k-[n/\overline{n}]}$ are uniquely determined and the deformation is perturbatively meromorphic of order two, the effective Planck mass being $\lambda_{\ell\text{eff}} = \lambda_\ell/\alpha^2 = 1/M_{P\text{eff}}^\ell$.

Here $\overline{s}_{\overline{n}}(\alpha) \sim \alpha^{Q_{\overline{n}}^{(2)}}$, so when $r_{\overline{n},\ell}^{(2)}$ is not ingeter perturbative meromorphy implies $\xi_{\overline{n}} = 0$, when $r_{\overline{n},\ell}^{(2)} = \overline{k} \in \mathbb{N}$ it is compulsory to introduce a new independent coupling at order $\overline{k}$ and when $r_{\overline{n},\ell}^{(2)} = -\overline{r}$ is a negative integer it is possible to add an arbitrary parameter at the price of increasing the order of perturbative meromorphy.

If $\beta_\alpha^{(1)} = 0$, $\gamma_{\overline{n}\ell}^{(1)} = \overline{n}\gamma_\ell^{(1)}$ and $\beta_\alpha^{(2)} = 0$ then the existence conditions become

$$\gamma_{\overline{n}\ell}^{(2)} \neq \overline{n}\gamma_\ell^{(2)} \,,$$

and so on. If $\beta_\alpha \equiv 0$, the procedure just described can be iterated until a k is found such that $\gamma_{\overline{n}\ell}^{(k)} - \overline{n}\gamma_\ell^{(k)} \neq 0$. Only when $\gamma_{\overline{n}\ell}^{(k)} - \overline{n}\gamma_\ell^{(k)} = 0$ for every k it is necessary to introduce a new parameter. If k_n denotes the minimum integer such that $\gamma_{n\ell}^{(k_n)} - n\gamma_\ell^{(k_n)} \neq 0$ and $\overline{k}$ denotes the maximum k_n, then the deformation is perturbatively meromorphic of order $\overline{k}$. The properties of finite and quasi-finite irrelevant deformations, which are precisely the case $\beta_\alpha \equiv 0$, are thus recovered.

In conclusion, the infinite reduction works in most models and its main properties are very general, although the details depend on the particular model. The existence conditions have the form (4.9), (4.15) or require the invertibility of matrices such as (4.17), whose entries are rational numbers divided by $\pi^{d/2}$, in even dimensions d. Generically, such conditions are violated only in sporadic cases. A sufficiently non-trivial renormalization mixing can make the infinite reduction work with a finite number of independent couplings. If this is not the case, the number of independent couplings can grow together with the order of the perturbative expansion, and the final theory can contain infinitely many independent couplings, but in general the growth is slow, in the sense that a reasonably small number of couplings are sufficient to make calculations up to very high orders. Thus, the infinite-reduction prescription enhances the predictive power considerably with respect to the usual formulation of non-renormalizable theories, where infinitely many independent couplings are present already at the tree level.

It is worth to mention that in the realm of renormalizable theories, the existence conditions for Zimmermann's reduction of couplings include the requirement that a certain discriminant be non-negative. A review and details are contained in appendix A, see formula (A.4). Several renormalizable theories are excluded by this restriction. In the absence of relevant parameters, the infinite reduction does not include constraints of this type. That is why the infinite reduction works in most models.

Finally, observe that the quantities that determine the invertibility conditions and the behavior of the solution for α small (i.e. Q_n, $\beta_\alpha^{(1)}$, $\gamma_n^{(1)}$, etc.) are scheme-independent. This proves that the infinite reduction is scheme-independent.

In odd dimensions, the main modification of the results derived above is that the one-loop coefficients of the beta functions and anomalous dimensions that appear in the invertibility conditions (4.9) are replaced by two-loop coefficients. Indeed, diagrams with an odd number of loops have no logarithmic divergences in odd dimensions. The other modifications follow straightforwardly from the arguments.

5. Interpretation of the infinite reduction

In this section I give an interpretation of the results derived so far, to better clarify the meaning of the infinite reduction.

Formulated in the ordinary way, a non-renormalizable theory contains infinitely many independent couplings, one for each essential, local, symmetric, scalar irrelevant operator constructed with the fields and their derivatives. In such a situation all non-renormalizable interactions are turned on and mixed, and the theory is predictive only as an effective field theory. The first step towards the construction of fundamental non-renormalizable theories is to "diagonalize" the non-renormalizable interactions, for example relating the irrelevant terms in a self-consistent and scheme-independent way. Renormalization-group invariance leads to equations (4.6), which do relate the irrelevant couplings to one another, but are not sufficient, by themselves, to reduce the number of couplings. Indeed, (4.6) are ingeneral differential equations, so their solutions contain one free independent parameter ξ for each coupling that is "reduced". Lucky situations are those in which the equations (4.6) are actually algebraic, which happens when the renormalizable subsector $\mathcal{R}$ is a conformal field theory. Moreover, if $\mathcal{R}$ does not contain relevant parameters, the beta functions of the irrelevant sector are linear in their own couplings and the solution generically exists, is unique and can be worked out iteratively.

When $\mathcal{R}$ is interacting and running, the arbitrary constants ξ generically multiply non-meromorphic functions of the marginal couplings α. Interactions that are not perturbatively meromorphic relatively to one another are, in some sense, "incommensurable". Any attempt to merge them produces violations of perturbative meromorphy that can be used to unmerge them back unambiguously. In practice, perturbative meromorphy classifies the fundamental non-renormalizable interactions and can be used to truly reduce the number of couplings. By means of renormalization and (relative) perturbative meromorphy, quantum field theory intrinsecally "knows" which interactions are which. The scheme-independence of the infinite reduction ensures that two observers that independently apply the reduction prescription arrive at the same conclusions.

These facts uncover the intrinsic nature of fundamental interactions. A local monomial $H(\varphi)$ in the fields and their derivatives does not provide a good description of a fundamental interaction $\Im$, because it is in general unstable under renormalization. The interaction can be stabilized under renormalization when the "head" $H(\varphi)$ is followed by a queue Q that runs coherently with it:

$$\Im(\alpha, \lambda, \varphi) = \lambda H(\varphi) + \sum_n \lambda_n(\alpha, \lambda) Q_n(\varphi)\,. \tag{5.1}$$

The queue is the sum of (generically infinitely many) local monomials $Q_n(\varphi)$ in the fields and their derivatives, multiplied by unambiguous functions $\lambda_n(\alpha, \lambda)$ of λ and α, that can be worked out recursively in n. When the invertibility conditions studied in the previous section are fulfilled, the queue is uniquely determined by perturbative meromorphy in α. In other cases new couplings $\overline{\lambda}$ are sporadically introduced along the way, guided by the reduction mechanism itself. The theory obtained deforming $\mathcal{R}$ with $\Im(\alpha, \lambda, \varphi)$ is renormalized by field redefinitions and renormalization constants for α, λ and eventually $\overline{\lambda}$.

The basis (5.1) "diagonalizes" the non-renormalizable interactions, and defines, for example, *the* Pauli deformation, that is to say the interaction whose head is the Pauli term, *the* four-fermion deformation (the interaction whose head is a four-fermion vertex), *the* Majorana-mass deformation, which is useful for physics beyond the Standard Model, the combination of some of them, and so on. Hopefully it will soon be possible to define the "Newton deformation", which encodes quantum gravity.

At the quantum level there exists one special basis (5.1) for the fundamental interactions, while classically all basis are equally good. Nevertheless, this is not a selection of theories, in general, because the non-renormalizable interactions $\Im$ are still infinitely many. The infinite reduction does not say which interactions are switched off and which ones are switched on in nature. In special situations the infinite reduction can also work as a selection, as it happens, for example, in three-dimensional quantum gravity coupled with matter (see sections 2 and 7). In more general situations there remains to find a physical criterion to select the right irrelevant deformation, or explain why no irrelevant deformation (that is to say the undeformed renormalizable theory $\mathcal{R}$) is better than any irrelevant deformation. Thus the infinite reduction is, in general, a classification of the non-renormalizable interactions, but not a selection. It is also the basic tool to address the selection issue and makes us hope that a better understanding of the problem of quantum gravity in four dimensions can be achieved also.

6. Infinite reduction of couplings around interacting fixed points

In the previous sections I have shown that a criterion for the infinite reduction is perturbative meromorphy around the free-field limit. In this section I study the infinite reduction in the neighborhood of an interacting fixed point. I consider theories whose renormalizable subsector $\mathcal{R}$ contains a single marginal coupling α and interpolates between UV and IR fixed points. For simplicity, I assume that one fixed point is free and the other one is interacting, since this is the more familiar situation and generalizations are straightforward. I show that another criterion for the infinite reduction is analyticity around the interacting fixed point. In general, the invertibility conditions are less restrictive than the one found in section 4 and the number of independent couplings of the final theory remains finite. Moreover, perturbative meromorphy around the free fixed point and analyticity around the interacting fixed point do not hold contemporarily, but only one at a time.

It is convenient to parametrize the beta function of α in the form

$$\beta_\alpha = \alpha^2 \, (\alpha_* - \alpha) \, B(\alpha) \, , \tag{6.1}$$

where $B(\alpha)$ is non-vanishing and analytic throughout the RG flow, with $B(0) = \beta_\alpha^{(1)}/\alpha_*$ and $B(\alpha_*) = -\beta'_*/\alpha_*^2$, $\beta_\alpha^{(1)}$ being the one-loop coefficient and β'_* being the slope of the beta function at the interacting fixed point. For definiteness, I assume $\alpha, \alpha_* \geq 0$ and $\beta_\alpha^{(1)}, \beta'_* \neq 0$ and that the anomalous dimensions $\gamma_n(\alpha)$ are regular and finite throughout the RG flow. When α is small, the anomalous dimensions are, generically, of order α, $\gamma_n(\alpha) = \gamma_n^{(1)} \alpha + \mathcal{O}(\alpha^2)$. Around the interacting fixed point, instead, they tend to constant values, $\gamma_n(\alpha) = \gamma_n^* + \mathcal{O}(\alpha_* - \alpha)$.

Consider an irrelevant deformation (4.4), with reduction relations (4.3). Once the reduction functions $f_k(\alpha) = \overline{f}_k(\alpha)$ are known for $k < n$, δ_n is a known function of α. Write $\overline{\delta}_n(\alpha) = \delta_n(\overline{f}_{k<n}(\alpha), \alpha)$. The solution of the RG consistency conditions (4.6) for f_n reads

$$f_n(\alpha, \xi) = \int_{c_n}^{\alpha} \mathrm{d}\alpha' \frac{\overline{\delta}_n(\alpha') \; s_n(\alpha, \alpha')}{\beta_\alpha(\alpha')} \, , \qquad s_n(\alpha, \alpha') = \frac{\overline{s}_n(\alpha)}{\overline{s}_n(\alpha')} \, , \tag{6.2}$$

where $\overline{s}_n(\alpha)$ is defined in (4.19) and c_n is the arbitrary integration constant, related in a simple way with the constants ξ_n used in section 4. Studying formula (4.19) around the fixed points it is immediately found that

$$\overline{s}_n(\alpha) = \alpha^{Q_n} (\alpha_* - \alpha)^{Q_n^*} U(\alpha) \, , \tag{6.3}$$

where

$$Q_n = \frac{\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}} \, , \qquad Q_n^* = \frac{\gamma_{n\ell}^* - n\gamma_\ell^*}{\beta'_*} \, ,$$

and $U(\alpha)$ is non-vanishing and analytic throughout the RG flow.

Now I prove that if the invertibility conditions

$$Q_n^* \notin \mathbb{N}, \qquad n > 1, \tag{6.4}$$

hold, the infinite reduction is uniquely determined by analyticity around the interacting fixed point. Assume, by induction, that the functions $\overline{f}_k(\alpha)$ with $k < n$ are unique and analytic for $\alpha \sim \alpha_*$. It if sufficient to show that there exists a unique choice of c_n such that also (6.2) is analytic for $\alpha \sim \alpha_*$.

The inductive assumption ensures that $\delta_n(\alpha) = \delta_n^* + \mathcal{O}(\alpha_* - \alpha)$ around the interacting fixed point, where δ_n^* is a numerical factor. Using (6.1) and (6.3) write (6.2) as

$$f_n(\alpha) = \overline{s}_n(\alpha) \int_{c_n}^{\alpha} \mathrm{d}\alpha' \frac{\delta_n(\alpha')U^{-1}(\alpha')B^{-1}(\alpha')}{(\alpha')^{2+Q_n}(\alpha_* - \alpha')^{1+Q_n^*}}\,. \tag{6.5}$$

The properties collected so far ensure that there exists an expansion

$$\alpha^{-2-Q_n}\delta_n(\alpha)U^{-1}(\alpha)B^{-1}(\alpha) = \sum_{k=0}^{\infty} a_{n,k}(\alpha_* - \alpha)^k\,.$$

Integrating (6.5) term-by-term it is immediate to prove that the most general solution (6.2) has the form

$$f_n(\alpha,\xi) = \overline{f}_n(\alpha) + \xi_n \overline{s}_n(\alpha)\,, \tag{6.6}$$

where

$$\overline{f}_n(\alpha) = \alpha^{Q_n} U(\alpha) \sum_{k=0}^{\infty} \frac{a_{n,k}(\alpha_* - \alpha)^k}{Q_n^* - k}$$

and ξ_n is a constant factor related with c_n.

If (6.4) hold, the function $\overline{f}_n(\alpha)$ is meaningful, and analytic, around the interacting fixed point. Instead, $\overline{s}_n(\alpha)$ is not analytic for $\alpha \sim \alpha_*$. Thus analyticity selects $\xi_n = 0$ and uniquely determines the infinite reduction, which reads

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi,\alpha] + \lambda_\ell \mathcal{O}_\ell(\varphi) + \sum_{n=2}^{\infty} \lambda_\ell^n \overline{f}_n(\alpha) \mathcal{O}_{n\ell}(\varphi)\,. \tag{6.7}$$

At the interacting fixed point

$$\overline{f}_n(\alpha) = -\frac{\delta_n^*}{\beta_*' Q_n^*} + \mathcal{O}(\alpha_* - \alpha) \tag{6.8}$$

and the deformed theory (6.7) tends to the quasi-finite theory

$$\mathcal{L}_{\mathrm{quasi-finite}}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi,\alpha_*] + \lambda_\ell \mathcal{O}_\ell(\varphi) - \sum_{n=2}^{\infty} \frac{\delta_n^*}{\gamma_n^* - n\gamma_1^*}\ \lambda_\ell^n\ \mathcal{O}_{n\ell}(\varphi)\,, \tag{6.9}$$

whose irrelevant couplings solve the algebraic quasi-finiteness equations

$$f_n(\alpha^*)\ \left(\gamma_n(\alpha^*) - n\gamma_1(\alpha^*)\right) + \delta_n(f(\alpha^*),\alpha^*) = 0\,,$$

obtained setting $\beta_\alpha = 0$ in (4.6). The existence conditions (6.4) collapse to just $\gamma^*_{n\ell} \neq n\gamma^*_\ell$, i.e. (3.6). Thus the irrelevant deformations of running theories selected by analyticity around the interacting fixed point are consistent with the finite and quasi-finite irrelevant deformations of interacting conformal field theories of sections 2 and 3.

Observe that the quantities that determine the invertibility conditions and the behavior of the solution around the fixed points (i.e. Q^*_n, δ^*_n, β'_*, γ^*_n, etc.) are scheme-independent, so the infinite reduction around the interacting fixed point is scheme-independent.

The invertibility conditions (6.4) are less restrictive than the invertibility conditions (4.9) for the existence of the infinite reduction around the free fixed point, because the numbers Q^*_n are not rational, in general. It is unlikely that (6.4) are violated for infinitely many ns, so in most models the analytic reduction around the interacting fixed point produces a theory whose divergences can be renormalized with a finite number of independent couplings, plus field redefinitions.

If (6.4) is violated for some $\overline{n}$, $Q^*_{\overline{n}} = \overline{k} \in \mathbb{N}$, a new coupling $\overline{\lambda}_{\overline{n}\ell}$ has to be introduced at order $\overline{k}$ in $\alpha - \alpha^*$, with a mechanism similar to the one explained in section 4. The reduction relations are extended preserving analyticity in α. Any new coupling $\overline{\lambda}_{\overline{n}\ell}$ introduced due to violations of (6.4) disappears in the limit $\alpha \to \alpha^*$ if $Q^*_{\overline{n}} \in \mathbb{N}_+$ and survives if $Q^*_n = 0$. Indeed, as noted above, the invertibility conditions become just $Q^*_n \neq 0$ in this limit.

If $Q^*_{\overline{n}}$ is a negative integer, then (6.6) is meromorphic around the interacting fixed point for $\xi_{\overline{n}} \neq 0$. Then an arbitrary parameter can be introduced at the price of relaxing analyticity to perturbative meromorphy. This case is analogous to the one discussed in section 4, see formula (4.26). The invertibility conditions, the reduction relations for $n > \overline{n}$, and the effective Planck mass, are modified following the instructions given in section 4.

Finally, observe that perturbative meromorphy around the free fixed point and analyticity around the interacting fixed point do not hold at the same time, in general. This fact can be easily proved integrating (6.2) exactly in the leading approximation $B(\alpha) = 1, \delta_n(\alpha) = \delta_n^{(1)}/\alpha^{n-2}$, $\gamma_n(\alpha) = \gamma_n^{(1)}\alpha$, etc. Indeed, there is no reason why the values of ξ_n that ensure analyticity around the interacting fixed point should coincide with the values of ξ_n that ensure perturbative meromorphy around the free fixed point. This property is a bit disappointing, but in the realm of non-renormalizable theories there is no physical reason to require a nice high-energy limit, so only the IR fixed point matters, free or interacting.

7. Applications

Some examples are useful to illustrare the arguments of the previous sections. I consider the Pauli deformation of massless QCD and quantum gravity coupled with matter in three spacetime dimensions. I also comment on the difficulties of four-dimensional quantum gravity.

Pauli deformation of massless QCD. As an example, consider the conformal window of massless QCD,

$$\mathcal{L} = \frac{1}{4\alpha}(F^a_{\mu\nu})^2 + \overline{\psi}\not{D}\psi\,,$$

with N_c colors and N_f flavors in the fundamental representation, in the limit where N_f, N_c are large but $N_f/N_c \lesssim 11/2$. The UV-fixed point is free, while the IR fixed point is interacting, but weakly coupled, so it can be reached perturbatively. The beta function reads

$$\beta_\alpha = -\frac{\Delta N_c}{24\pi^2}\alpha^2 + \frac{25N_c^2}{(4\pi)^4}\alpha^3 + \alpha\sum_{n=3}^{\infty} c_n\,(\alpha N_c)^n\,, \tag{7.1}$$

where $\Delta \equiv 11-2N_f/N_c$, $0 < \Delta \ll 1$ and the c_ns are unspecified numerical coefficients. The first two contributions of the beta function have opposite signs and the first contribution is arbitrarily small. This ensures that, expanding in powers of Δ, the beta function has a second zero for

$$\frac{\alpha_* N_c}{16\pi^2} = \frac{2}{75}\Delta + \mathcal{O}(\Delta^2)\,, \tag{7.2}$$

which is the IR fixed point of the RG flow.

The Pauli deformation [2] is the irrelevant deformation with head

$$\lambda_1\; F^a_{\mu\nu}\;\overline{\psi}T^a\sigma_{\mu\nu}\psi$$

and has level one. The queue begins with operators of level two. There are 10 four-fermion operators and the F^3 term

$$\frac{\lambda_2}{3!}\; f^{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}\,.$$

For simplicity, I consider only this level-2 term, because the argument is completely general and the extension to four-fermion operators is straightforward. The one-loop beta functions are [2]

$$\beta_{\lambda_1} = \frac{\alpha N_c}{16\pi^2}\lambda_1\,, \qquad \beta_{\lambda_2} = \frac{3N_c\alpha}{4\pi^2}\lambda_2 - \frac{N_f}{4\pi^2}\lambda_1^2\,,$$

so in this case

$$Q_2 = \frac{\gamma_2^{(1)} - 2\gamma_1^{(1)}}{\beta_\alpha^{(1)}} = -\frac{15}{\Delta} = -Q_2^*\,, \qquad \lambda_2 = \overline{f}_2(\alpha)\lambda_1^2 = \frac{11}{5\alpha}\lambda_1^2\,(1+\mathcal{O}(\alpha,\Delta))\,.$$

Similar formulas hold for Q_n, $n > 2$. Note that in this approximation Q_n and the Q_n^* are related in a simple way, namely $Q_n = -Q_n^*$. This happens because the interacting fixed point is weakly coupled.

The invertibility conditions (4.15) and (6.4) for $n = 2$ read

$$1 - \frac{15}{\Delta} \notin \mathbb{N}\,, \qquad \frac{15}{\Delta} \notin \mathbb{N}\,, \tag{7.3}$$

around the free-field limit and the interacting fixed point, respectively. Similar conditions are expected for $n > 2$:

$$Q_n + q_n = \frac{b_n(\Delta)}{\Delta} \notin \mathbb{N}\,, \qquad Q_n^* = \frac{b_n'(\Delta)}{\Delta} \notin \mathbb{N}\,, \tag{7.4}$$

q_n being Δ-independent quantities depending on the level and the number of legs of $\mathcal{O}_{n\ell}(\varphi)$, $b_n(\Delta)$ and $b_n'(\Delta)$ being rational numbers with smooth $\Delta \to 0$ limits $b_n(0) = -b_n'(0)$. A non-trivial renormalization mixing can only make the invertibility conditions less restrictive, so I proceed assuming the worst case, which is (7.4). Since Δ is rational and arbitrarily small, while Δ tends to zero the numbers $Q_n + n - 1$ and Q_n^* cross, among the others, also natural integer values and so violate the conditions (7.4). However, if $b_n(0) \neq 0$ ($b_n'(0) \neq 0$), when $b_n(\Delta)/\Delta \in \mathbb{N}$ ($b_n'(\Delta)/\Delta \in \mathbb{N}$), the new parameters appear at orders $b_n(\Delta)/\Delta$ ($b_n'(\Delta)/\Delta$) that are arbitrarily high for $\Delta \ll 1$. Therefore, the effects of the violations of (7.4) are negligible in the limit discussed here, and the Pauli deformation is determined unambiguously under the sole conditions $b_n(0) \neq 0$, $b_n'(0) \neq 0$, which are equivalent to $\gamma_{n\ell}^{(1)} \neq n\gamma_\ell^{(1)}$ at $\Delta = 0$.

Finally, in the limit where the momenta and $1/\lambda_1$ are much smaller than the dynamical scale μ the deformed theory tends to the quasi-finite theory studied in ref. [2]. Indeed, using (6.8) it is immediately found that

$$\overline{f}_2(\alpha_*) = \frac{165 N_c}{32\Delta\pi^2},$$

in agreement with [2]. Observe that the conditions for the existence of the irrelevant deformation of the RG flow (namely $\gamma_{n\ell}^{(1)} \neq n\gamma_\ell^{(1)}$ at $\Delta = 0$) coincide with the conditions for the existence of the quasi-finite deformation of the IR fixed point.

Three-dimensional quantum gravity coupled with matter. The results of this paper can be applied also to three-dimensional quantum gravity coupled with running matter, and generalize the results of ref. [1], where the matter sector was assumed to be conformal.

If the invertibility conditions are fulfilled, the theory is unique. If new parameters appear along the way, but they are finitely many, the theory is still predictive. Finally, if infinitely many parameters are turned on by the infinite reduction, they are generically expected to appear at increasingly high orders. Then the predictivity of the theory is of the type discussed in section 4: it is still possible to make calculations up to very high orders with a relatively small number of couplings.

Four-dimensional quantum gravity. Applications to four-dimensional quantum gravity, instead, demand further insight. The renormalizable subsector of gravity is not interacting, so the infinite reduction does not apply. Equivalently, the effective Planck mass $M_{P\text{eff}}$ is zero, so the perturbative regime $E \ll M_{P\text{eff}}$ is empty. The reason why three-dimensional quantum gravity is exceptional is that, although the renormalizable subsector of gravity is free, all irrelevant operators constructed with the Riemann tensor and their derivatives are trivial, in the sense that they can be converted into matter operators using the field equations. Then, to have a non-trivial effective Planck mass it is sufficient to have an interacting matter sector.

8. Irrelevant deformations with several marginal running couplings

To complete the investigation of this paper, I study the irrelevant deformations of running renormalizable theories $\mathcal{R}$ containing more than one independent marginal coupling. The purpose is to show that the infinite reduction is free of some difficulties that are present in

Zimmermann's analytic reduction (see appendix A for details) and better appreciate some other properties.

I study the behavior of the reduction relations in a neighborhood of the free fixed point, focusing on the leading-log approximation, for which the one-loop coefficients of the beta functions and anomalous dimensions suffice. Consider a renormalizable theory with couplings α_1 and α_2 and one-loop beta functions

$$\beta_{\alpha_1} = \beta_1 \alpha_1^2 \,, \qquad \beta_{\alpha_2} = a\alpha_1^2 + b\alpha_1\alpha_2 + c\alpha_2^2 \,,$$

where β_1, a, b, c are unspecified numerical factors. For intermediate purposes, it is useful to "reduce" the marginal sector to a unique running constant, say α_1, plus a finite arbitrary parameter c_1, following Zimmermann's method. Solve the RG consistency equations

$$\frac{\mathrm{d}\alpha_2}{\mathrm{d}\alpha_1} = \frac{\beta_{\alpha_2}(\alpha_1, \alpha_2)}{\beta_{\alpha_1}(\alpha_1, \alpha_2)} \,.$$

The solution reads

$$\widetilde{\alpha}_2(\alpha_1, c_1) = -\frac{\alpha_1}{2c}\left[b - \beta_1 + s\frac{1 + (\alpha_1/c_1)^{-s/\beta_1}}{1 - (\alpha_1/c_1)^{-s/\beta_1}}\right] , \tag{8.1}$$

where $s = \sqrt{(b - \beta_1)^2 - 4ac}$. The quantity s can be complex, but this does not cause problems here, because (8.1) is used only for intermediate purposes. In the end c_1 is eliminated in favor of α_2 using the inverse of (8.1):

$$c_1 = \alpha_1 z^{-\beta_1/s} \,, \qquad \mathrm{where} z = \frac{2c\alpha_2 + \alpha_1(b - \beta_1 - s)}{2c\alpha_2 + \alpha_1(b - \beta_1 + s)} \,. \tag{8.2}$$

Observe that the function $\alpha_1 z^{-\beta_1/s}$ is constant along the RG flow. The noticeable modular combination z plays an important role throughout the discussion.

Consider an irrelevant deformation of level ℓ, with coupling λ_ℓ. The first term of the queue is multiplied by the coupling $\lambda_{2\ell}$. For small $\alpha_{1,2}$, the lowest-order beta functions of λ_ℓ and $\lambda_{2\ell}$ have generically the forms

$$\beta_{\lambda_\ell} = \lambda_\ell(d\alpha_1 + e\alpha_2) \,, \qquad \beta_{\lambda_{2\ell}} = \lambda_{2\ell}(f\alpha_1 + g\alpha_2) + h\lambda_\ell^2 \,, \tag{8.3}$$

where d, e, f, g, h are unspecified numerical factors. Search for a reduction relation of the form

$$\lambda_{2\ell} = f_2(\alpha_1, \alpha_2)\lambda_\ell^2 \,.$$

Differentiating this expression and using (8.3), the equation obeyed by f_2 reads

$$\frac{\mathrm{d}f_2}{\mathrm{d}\ln\mu} = h - 2f_2\left(\widetilde{d}\alpha_1 + \widetilde{e}\alpha_2\right) , \tag{8.4}$$

where $\widetilde{d} = d - f/2$ and $\widetilde{e} = e - g/2$. Now, (8.4) is one differential equation for a function of two variables, so the most general solution contains an arbitrary function of one variable (see below). Call

$$\widetilde{f}_2(\alpha_1, c_1) = f_2\left(\alpha_1, \widetilde{\alpha}_2(\alpha_1, c_1)\right) \tag{8.5}$$

the solution of the equation

$$\beta_1\alpha_1^2\frac{\mathrm{d}\widetilde{f}_2(\alpha_1,c_1)}{\mathrm{d}\alpha_1}+2\widetilde{f}_2(\alpha_1,c_1)\left(\widetilde{d}\alpha_1+\widetilde{e}\widetilde{\alpha}_2(\alpha_1,c_1)\right)=h\,, \tag{8.6}$$

which is obtained inserting (8.1) into (8.4). The solution of (8.5) depends on c_1 and a further arbitrary constant c_2. Eliminating c_1 with the help of (8.2), the solution reads

$$f_2(\alpha_1,\alpha_2)=\overline{f}_2(\alpha_1,\alpha_2)+c_2\overline{s}_2(\alpha_1,\alpha_2)\,, \tag{8.7}$$

where

$$\overline{f}_2(\alpha_1,\alpha_2)=\frac{2h\;{}_2F_1[1,\gamma-2\widetilde{e}/c,\gamma,z]}{(2c\alpha_2+\alpha_1(s+b-\beta_1))\,(\gamma-1)}\,,\qquad\gamma=\frac{1}{cs}\Big(c(2\widetilde{d}-\beta_1+s)+\widetilde{e}(s-b+\beta_1)\Big)\,, \tag{8.8}$$

and

$$\overline{s}_2(\alpha_1,\alpha_2)=z^{-\delta}\;(2c\alpha_2+\alpha_1(s+b-\beta_1))^{-2\widetilde{e}/c}\,,\qquad\delta=\frac{\widetilde{e}(s-b-\beta_1)+2c\widetilde{d}}{cs}\,. \tag{8.9}$$

Now, c_2 is an arbitrary constant of the RG equation (8.4). This means that c_2 is constant only along the RG flow, but can otherwise depend on α_1 and α_2. The function of α_1 and α_2 that is constant along the RG flow is given in eq. (8.2), so c_2 is an arbitrary function k_2 of $\alpha_1 z^{-\beta_1/s}$. In conclusion, the most general solution of (8.4) reads

$$f_2(\alpha_1,\alpha_2)=\overline{f}_2(\alpha_1,\alpha_2)+k_2(\alpha_1 z^{-\beta_1/s})\;\overline{s}_2(\alpha_1,\alpha_2)\,.$$

The remarkable points are $z=0,1,\infty$, i.e.

$$\alpha_2+\frac{\alpha_1}{2c}(b-\beta_1-s)=0\,,\qquad\alpha_1=0\,,\qquad\alpha_2+\frac{\alpha_1}{2c}(b-\beta_1+s)=0\,, \tag{8.10}$$

respectively. These are the combinations of couplings that vanish together with their own beta functions at the leading-log level. Along these lines a subsector of the theory is practically at a fixed point, in the given approximation. Therefore, the reduction should be perturbatively meromorphic, or analytic, in the neighborhood of such lines. However, formulas (8.7), (8.8) and (8.9) show that perturbative meromorphy can be imposed only in the neighborhood of one line (8.10) at a time, not around all of them contemporarily. The situation is similar to the one discussed in section 6, where it was observed that perturbative meromorphy around the free-field limit and analyticity around the interacting fixed point mutually exclude each other. Once the line of perturbative meromorphy is chosen, the function k_2 is uniquely determined: in the order (8.10), the results are

$$\begin{aligned}
\overline{f}_2(\alpha_1,\alpha_2)&=\frac{2h\;{}_2F_1[1,\gamma-2\widetilde{e}/c,\gamma,z]}{(2c\alpha_2+\alpha_1(b-\beta_1+s))\,(\gamma-1)}\,,&&1-\gamma\notin\mathbb{N}\,,\\
\overline{f}'_2(\alpha_1,\alpha_2)&=\frac{2h\;{}_2F_1[1,\gamma-2\widetilde{e}/c,2-2\widetilde{e}/c,1-z]}{(2c\alpha_2+\alpha_1(b-\beta_1+s))\,(2\widetilde{e}/c-1)}\,,&&2\frac{\widetilde{e}}{c}-1\notin\mathbb{N}\,,\\
\overline{f}''_2(\alpha_1,\alpha_2)&=-\frac{2h\;{}_2F_1[1,2-\gamma,2-\gamma+2\widetilde{e}/c,1/z]}{(2c\alpha_2+\alpha_1(b-\beta_1-s))\,(\gamma-2\widetilde{e}/c-1)}\,,&&\gamma-2\frac{\widetilde{e}}{c}-1\notin\mathbb{N}\,.
\end{aligned}$$

To the right the respective existence conditions are reported.

The other terms of the queue are worked out similarly. As before, the invertibility conditions involve only scheme-independent coefficients and new independent couplings can sporadically appear at high orders.

9. Conclusions

I have studied methods to classify the non-renormalizable interactions and criteria to remove the infinite arbitrariness of non-renormalizable theories, taking inspiration from recent constructions of finite and quasi-finite irrelevant deformations of interacting conformal field theories. I have considered non-renormalizable theories whose renormalizable subsector $\mathcal{R}$ is fully interacting, running, with one or more marginal couplings. Relevant couplings can be added perturbatively to the constructions of this paper.

An irrelevant deformation is made of a head and a queue that runs coherently with the head. The head is the lowest-level irrelevant term, multiplied by an independent coupling λ_ℓ. The queue is made of an infinite number of irrelevant terms with higher dimensionalities in units of mass. "Reduction" relations express the couplings of the queue as functions of λ_ℓ and the marginal couplings α of $\mathcal{R}$. The reduction relations are polynomial in λ_ℓ. The α -dependence is determined by consistency with the renormalization group and one of the following scheme-independent prescriptions: (i) perturbative meromorphy around a free fixed point of $\mathcal{R}$, or (ii) analyticity around an interacting fixed point of $\mathcal{R}$. In general, it is not possible to have both at the same time. The infinite reduction works when certain invertibility conditions are fulfilled. In the case of violations, new independent couplings $\overline{\lambda}_{\rm new}$ are introduced along the way. The divergences of a theory reduced with these criteria are reabsorbed into renormalization constants for α, λ_ℓ and eventually $\overline{\lambda}_{\rm new}$, plus field redefinitions.

With prescription (i) the number of independent couplings remains finite or grows together with the order of the expansion. It remains finite if the irrelevant operators have a sufficiently non-trivial renormalization mixing. When the number of couplings grows together with the order of the expansion, the growth is in general so slow that a reasonably small number of couplings are sufficient to make predictions up to very high orders. With prescription (ii) the number of couplings generically remains finite.

The infinite reduction does not determine which non-renormalizable interactions are switched on and off in nature, but is the basic tool to classify the non-renormalizable interactions and address the search for selective criteria. In my opinion the theories costructed with the infinite-reduction prescription are as fundamental as the usual renormalizable theories.

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A. Zimmermann's reduction of couplings

In this section I review the main properties of Zimmermann's "reduction of couplings" [4]. I also describe some difficulties of the analytic prescription and emphasize the properties of perturbative meromorphy. It is convenient to have a concrete example in mind, such as massless scalar electrodynamics,

$$\mathcal{L} = \frac{1}{4\alpha}F_{\mu\nu}^2 + |D_\mu\varphi|^2 + \frac{\lambda}{4}(\overline{\varphi}\varphi)^2\,, \tag{A.1}$$

where $D_\mu\varphi = \partial_\mu\varphi + iA_\mu\varphi$. The reduction is a function $\lambda(\alpha)$ that relates the two couplings. Consistency with the renormalization group gives the differential equation

$$\frac{\mathrm{d}\lambda(\alpha)}{\mathrm{d}\alpha} = \frac{\beta_\lambda(\lambda(\alpha),\alpha)}{\beta_\alpha(\lambda(\alpha),\alpha)}\,, \tag{A.2}$$

that determines the solution $\lambda(\alpha)$ up to an arbitrary constant $\overline{\lambda}$, the initial condition. The structures of the beta functions are

$$\begin{aligned}\beta_\alpha &= \alpha\left(\beta_1\alpha + \beta_{21}\lambda^2 + \beta_{22}\lambda\alpha + \beta_{23}\alpha^2 + \cdots\right),\\ \beta_\lambda &= a_1\alpha^2 + a_2\lambda\alpha + a_3\lambda^2 + b_1\lambda^3 + b_2\lambda^2\alpha + b_3\lambda\alpha^2 + b_4\alpha^3 + \cdots\,.\end{aligned} \tag{A.3}$$

Assume $\beta_1 \neq 0$. If

$$\Delta \equiv (a_2-\beta_1)^2 - 4a_1a_3 \geq 0\,, \tag{A.4}$$

and

$$r_\pm \equiv \pm\frac{s}{\beta_1} - 1 \notin \mathbb{N}\,, \tag{A.5}$$

where s is the positive square root of Δ, then the equations (A.2) are solved by the expansions

$$\lambda_\pm(\alpha,d) = \sum_{k=1}^{\infty} c_{\pm k}\alpha^k + \sum_{m,n=1}^{\infty} d_{\pm mn}d^n\alpha^{m\pm ns/\beta_1}\,, \tag{A.6}$$

where

$$c_{\pm 1} = \frac{1}{2a_3}\left(\beta_1 - a_2 \pm s\right), \qquad d_{\pm 11} = 1\,,$$

and the coefficients $c_{\pm k}$ and $d_{\pm mn}$ are unambiguous calculable numbers, while d is an arbitrary parameter. If $\beta_1 > 0$ the meaningful expansions are $\lambda_+(\alpha,d)$ and $\lambda_-(\alpha,0)$, if $\beta_1 < 0$ they are $\lambda_+(\alpha,0)$ and $\lambda_-(\alpha,d)$.

The condition (A.4) follows from the reality of λ in the expansion (A.6), while (A.5) are derived inserting (A.6) in (A.3) and (A.2). The coefficients $c_{\pm k}$ have expressions

$$c_{\pm k} = \frac{P_{\pm k}(c_{\pm 1}, a, b, \beta, \ldots)}{\prod_{j=2}^{k}\left[(1-j)\beta_1 \pm s\right]}\,, \qquad k > 1\,, \tag{A.7}$$

where $P_{\pm k}$ are polynomials that in general do not vanish when the denominator vanishes.

Clearly, the expansions $\lambda_\pm(\alpha,d)$ (A.6) are just different expansions of the same function $\lambda(\alpha,d)$, because for every value of d the solution is unique.

The condition (A.4) is quite restrictive, and excludes a good amount of models. On the other hand, when (A.4) holds, s is generically an irrational number and (A.5) is automatically satisfied. Therefore in Zimmermann's reduction the crucial existence condition is (A.4).

As long as d is arbitrary, there is no true reduction. The d-ambiguity can be eliminated demanding that the reduction be analytic. When (A.4) and (A.5) hold, both expansions

$\lambda_\pm(\alpha, d)$ are generically non-analytic at $d \neq 0$. Therefore analyticity implies $d = 0$ and gives two distinct unambiguous solutions

$$\lambda_\pm(\alpha) \equiv \lambda_\pm(\alpha, 0) = \sum_{n=1}^{\infty} c_{\pm k}\alpha^k . \tag{A.8}$$

On the other hand, if (A.4) holds, but (A.5) does not, then analyticity is violated by logarithms, which signal the presence of the other independent coupling. In this case the reduction is ineffective.

Now I compare Zimmermann's reduction with the infinite reduction. If the renormalizable aubsector $\mathcal{R}$ does not contain relevant couplings, the beta function of an irrelevant coupling λ is linear in λ. Then the infinite reduction has existence conditions of type (A.5), one for every term of the queue, but no existence condition of type (A.4). This is a lucky situation, since it would be hopeless to satisfy infinitely many reality conditions such as (A.4). On the other hand, the conditions of type (A.5) become (4.9) and involve only rational numbers. It is not unfrequent that some of these rational numbers coincide with natural numbers $\overline{k}$. For every such "coincidence" a new independent coupling is introduced at order $\overline{k}$.

A.1 Difficulties of Zimmermann's approach with two or more reduced couplings

Consider a generic renormalizable theory with three marginal couplings, α_1, α_2 and λ, with one-loop beta functions

$$\begin{aligned} \beta_{\alpha_1} &= \beta_1 \alpha_1^2 , \qquad \beta_{\alpha_2} = a\alpha_1^2 + b\alpha_1\alpha_2 + c\alpha_2^2 , \\ \beta_\lambda &= f\alpha_1^2 + g\alpha_1\alpha_2 + h\alpha_2^2 + \lambda(d\alpha_1 + e\alpha_2) + l\lambda^2 , \end{aligned}$$

and seek for an analytic reduction

$$\lambda(\alpha_1, \alpha_2) = c_1\alpha_1 + d_1\alpha_2 + c_2\alpha_1^2 + d_2\alpha_1\alpha_2 + e_2\alpha_2^2 + \cdots , \tag{A.9}$$

leaving two independent couplings and eliminating the third one. Differentiating (A.9) and matching the coefficients of α_1^2, $\alpha_1\alpha_2$ and α_2^2 with β_λ, the following equations are obtained:

$$\begin{aligned} c_1\beta_1 + d_1 a &= f + c_1 d + l c_1^2 , \\ d_1 b &= g + c_1 e + d_1 d + 2l c_1 d_1 , \\ d_1 c &= h + d_1 e + l d_1^2 . \end{aligned}$$

These are three (generically independent) equations for the two unknowns c_1 and d_1. The mismatch between the number of unknows and the number of equations has the following explanation. The expansion (A.9) is made of a sum of polynomials of degrees $n = 1, 2, \ldots$ in α_1 and α_2. The polynomial of degree n contributing to $\lambda(\alpha_1, \alpha_2)$ contains $n+1$ monomials and therefore $n+1$ unknown coefficients. After differentiation, these unknowns contribute to polynomials of higher orders, at least $n+2$, in the RG consistency conditions, thereby they appear in at least $n+2$ equations. Therefore the problem has, in general, no solutions [6].

This means that the analyticity requirement is too strong. The problems are avoided as explained in section 8.

B. Perturbative meromorphy and infinite reduction

In this appendix I define the notion of perturbative meromorphy and study some of its properties. A function $f(\lambda,\alpha)$ is said to be perturbatively meromorphic in α with respect to λ if it is analytic in λ and admits an expansion

$$f(\lambda,\alpha)=g(\alpha)+\sum_{n=1}^{\infty}c_n(\alpha)\lambda^n\,,$$

such that the functions $g(\alpha),c_n(\alpha)$ are meromorphic in α and the $c_n(\alpha)$s have at most poles of order $pn-q$, where p and q are non-negative. Assume for definiteness that the poles are in $\alpha=0$. Then it is clear that if $\lambda_{\text{eff}}=\lambda\alpha^{-p}$ the function

$$\overline{f}(\lambda_{\text{eff}},\alpha)=f(\lambda_{\text{eff}}\alpha^p,\alpha)=g(\alpha)+\alpha^q\sum_{n=0}^{\infty}\overline{c}_n(\alpha)\lambda_{\text{eff}}^n\,,$$

is analytic in λ_{eff} and meromorphic in α, since the $\overline{c}_n(\alpha)$s are regular.

For example, renormalizable quantum field theory is perturbatively meromorphic in ε with respect to $\hbar$ or the marginal couplings α, in the sense that the renormalization constants admit expansions of the form

$$\sum_{L=0}^{\infty}\sum_{k=0}^{L}c_{L,k}\left(\frac{1}{\varepsilon}\right)^k\alpha^L+\text{evanescent}=\sum_{L=0}^{\infty}c_L(\varepsilon)\alpha^L$$

where L is the number of loops and $c_L(\varepsilon)$ has at most a pole of order L. Here $p=1$ and $q=0$.

The infinite reduction is perturbatively meromorphic in the marginal couplings α with respect to the irrelevant couplings λ_ℓ. When $\alpha\sim 0$

$$\mathcal{L}[\varphi]\sim\mathcal{L}_{\mathcal{R}}[\varphi,\alpha]+\lambda_\ell\mathcal{O}_\ell(\varphi)+\sum_{n=2}^{\infty}\frac{c_n\alpha^{k_n}}{\alpha^{pn-q}}\lambda_\ell^n\ \mathcal{O}_{n\ell}(\varphi)\tag{B.1}$$

$$=\mathcal{L}_{\mathcal{R}}[\varphi,\alpha]+\alpha^p\lambda_{\ell\text{eff}}\mathcal{O}_\ell(\varphi)+\alpha^q\sum_{n=2}^{\infty}c_n\alpha^{k_n}\lambda_{\ell\text{eff}}^n\ \mathcal{O}_{n\ell}(\varphi)\,,\tag{B.2}$$

where $\lambda_{\ell\text{eff}}=\lambda_\ell\alpha^{-p}$, c_n are constants and k_n are non-negative integers. Normally, in the absence of three-leg vertices $p=q\geq 1$. In the presence of three-leg vertices the elementary marginal coupling is $g=\alpha^{1/2}$, which is equivalent to say $p\geq q=1/2$. The number q is the *order* of the irrelevant deformation. The order is always positive, which emphasizes that it is necessary to have an interacting renormalizable subsector to build the irrelevant deformations.

Observe that marginal and irrelevant deformations do not commute. It is possible to switch the irrelevant deformation off, keeping $\alpha\neq 0$, but it is impossible to switch the marginal deformation off keeping a non-trivial irrelevant sector in the limit $\alpha\to 0$. This is also the meaning of perturbative meromorphy: in some sense, the true head of the irrelevant deformation is the operator of infinite dimensionality.

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Now I prove some theorems that are useful in the paper. Observe that for each level n that fulfills the invertibility conditions, (4.9) or their appropriate generalizations, the differential operator

$$D_n \equiv \beta_\alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} - \gamma_{n\ell}(\alpha) + n\gamma_\ell(\alpha) \tag{B.3}$$

appearing in the RG consistency conditions (4.6) is $\mathcal{O}(\alpha)$ and its order $\mathcal{O}(\alpha)$ is non-vanishing. Then (B.3) can be freely inverted and $D_n^{-1} = \mathcal{O}(\alpha^{-1})$.

Theorem 1. *Suppose that $q \geq 1$, $2p \geq q+1$, that the renormalization structures (B.1), (B.2) are stable under renormalization up to the level $\overline{n}$ and that each level $> \overline{n}$ fulfills the invertibility conditions. Then the structures (B.1), (B.2) are stable under renormalization.*

Proof. Consider (B.2) and the beta functions (4.2) $\beta_{n\ell} = \gamma_{n\ell}\lambda_{n\ell} + \delta_{n\ell}$, $n > \overline{n}$. Since $\delta_{n\ell}$ is at least quadratic in the irrelevant couplings and in the notation (B.2) each of coupling carries at least a power α^s, $s = \min(p, q)$, then $\delta_{n\ell} \sim \alpha^{2s}\lambda_{\ell\mathrm{eff}}^n$ at worst, so, using $D_n^{-1} = \mathcal{O}(\alpha^{-1})$, $\lambda_{n\ell} \sim \alpha^{2s-1}\lambda_{\ell\mathrm{eff}}^n = \alpha^q\alpha^{2s-q-1}\lambda_{\ell\mathrm{eff}}^n$, which is compatible with (B.2), since $2s - q - 1 \geq 0$.

Theorem 2. *Suppose that $\lambda_{k\ell}$ behaves at worst as*

$$\lambda_{k\ell} \sim \frac{\lambda_\ell^k}{\alpha^{q(k-1)}}, \tag{B.4}$$

for $1 < k < n$, with $q \geq 1$, and that the invertibility conditions are fulfilled for $k \geq n$. Then the irrelevant deformation is perturbatively meromorphic of order q.

Proof. By induction, it is sufficient to prove (B.4) for $k = n$. Consider again the beta functions (4.2) $\beta_{n\ell} = \gamma_{n\ell}\lambda_{n\ell} + \delta_{n\ell}$. Since $\delta_{n\ell}$ depends on the $\lambda_{k\ell}$ with $k < n$, $\delta_{n\ell} \sim \prod_{k<n} {\lambda_{k\ell}}^{n_k}(1+\mathcal{O}(\alpha))$, with $\sum_{k<n} kn_k = n$, where n_k are non-negative integers. Moreover, $m \equiv \sum_{k<n} n_k \geq 2$, since $\delta_{n\ell}$ is at least quadratic. Therefore for small α, using $D_n^{-1} = \mathcal{O}(\alpha^{-1})$,

$$\lambda_{n\ell} \sim \frac{\delta_{n\ell}}{\alpha} \sim \frac{\lambda_\ell^n}{\alpha}\prod_{k<n}\left(\frac{1}{\alpha^{q(k-1)}}\right)^{n_k} = \frac{\lambda_\ell^n}{\alpha^{q(n-m)+1}} \leq \frac{\lambda_\ell^n}{\alpha^{q(n-1)}}. \tag{B.5}$$

Theorem 3. *Suppose that $\lambda_{k\ell}$ behaves at worst as*

$$\lambda_{k\ell} \sim \frac{\lambda_\ell^k}{\alpha^{q(k-1)}}, \tag{B.6}$$

with $q \geq 1$, for $k < \overline{n}$, that $\lambda_{\overline{n}\ell}$ has a more singular behavior

$$\lambda_{\overline{n}\ell} \sim \frac{\lambda_\ell^{\overline{n}}}{\alpha^{q(\overline{n}-1)+\overline{r}}}, \tag{B.7}$$

with $\overline{r} > 0$, and that the invertibility conditions are fulfilled for $k > \overline{n}$. Then the behavior of $\lambda_{k\ell}$ for generic k is at worst

$$\lambda_{k\ell} \sim \frac{\lambda_\ell^k}{\alpha^{q(k-1)+\overline{r}[k/\overline{n}]}} \tag{B.8}$$

and the multiple irrelevant deformation is perturbatively meromorphic of order

$$\overline{q} = q + \left[\frac{\overline{r}}{\overline{n}-1}\right]_+ , \tag{B.9}$$

$[x]_+$ *denoting the minimum integer* $\geq x$*. Moreover, the associated renormalization structure (B.2) with* $p, q \to \overline{q}$ *is stable under renormalization.*

Proof. Formula (B.8) certainly holds for $k \leq \overline{n}$. By induction, assuming (B.8) for $k < n$, with $n > \overline{n}$, it is sufficient to prove (B.8) for $k = n$. Indeed, using the same notation as in the proof of theorem 2,

$$\lambda_{n\ell} \sim \frac{\delta_{n\ell}}{\alpha} \sim \frac{\lambda_\ell^n}{\alpha} \prod_{k<n} \left(\frac{1}{\alpha^{q(k-1)+\overline{r}[k/\overline{n}]}}\right)^{n_k} = \frac{\lambda_\ell^n}{\alpha^{q(n-m)+1+\overline{r}\sum_{k<n} n_k [k/\overline{n}]}} \leq \frac{\lambda_\ell^n}{\alpha^{q(n-1)+\overline{r}[n/\overline{n}]}} .$$

The last inequality follows from

$$q(m-1) \geq 1 , \qquad \left[\frac{n}{\overline{n}}\right] \geq \sum_{k<n} n_k \left[\frac{k}{\overline{n}}\right] .$$

Moreover, (B.8) implies

$$\lambda_{n\ell} \leq \frac{\lambda_\ell^n}{\alpha^{\overline{q}(n-1)}} , \tag{B.10}$$

for every n. Indeed, for $n < \overline{n}$ (B.10) follows from $\overline{q} \geq q$. For $n = \overline{n}$ (B.10) follows from $[x]_+ \geq x$. For $n > \overline{n}$ (B.10) follows from

$$\overline{r}\left[\frac{n}{\overline{n}}\right] \leq \overline{r}\frac{n}{\overline{n}} < \overline{r}\frac{n-1}{\overline{n}-1} \leq \left[\frac{\overline{r}}{\overline{n}-1}\right]_+ (n-1).$$

So, the irrelevant deformation is perturbatively meromorphic of order $\overline{q}$. Finally, since $\overline{q} \geq 1$, by theorem 1 the associated renormalization structure is stable.

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